Commonality Analysis for a Family of Material Models

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Table of Units

Throughout this document consistent units are employed and a consistent notation is used. The unit system adopted is the "MLtT" dimension system, where M is the dimension of mass, L is length, t is time and T is temperature. This system corresponds nicely with the SI (Système International d'Unités), or modern metric, system, which uses units of kilogram (kg), meter (m), second (s) and Kelvin (K) for M, L, t and T, respectively. By leaving the units in this form any unit system can be adopted in the future, as long as the choice of specific units is consistent between different quantities. In addition to the basic units, several derived units are employed as described below. For each unit the symbol is given followed by a description of the unit with the SI equivalent in parentheses.

L	- length (metre, m)
M	- mass (kilogram, kg)
t	- time (second, s)
T	- temperature (Kelvin, K)
ForceU	- force, which has units of $\mathrm{M \cdot L \cdot t^{-2}}$ (Newton, $\mathrm{N = kg \cdot m \cdot s^{-2}}$)
StressU	- stress, which has units of $\mathrm{L^{-1}Mt^{-2}}$, or $\mathrm{ForceU/L^2}$ (Pascal, $\mathrm{Pa = N/m^2}$)
angleU	- radian (rad)

Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made with the goal of being consistent with the existing literature. Accomplishing this goal requires that some symbols are used for multiple purposes. Where this is the case, all possible meanings will be listed, with each on a separate line. When the symbol is later used in the body of the document, it will be defined on its first occurrence. In the other instances where the symbol occurs, its meaning should be clear from the context. The units are listed in two sets of brackets following the definition of the symbol. The first set of brackets shows the MLtT dimension system and the second set of brackets shows the equivalent SI units. In the cases where the symbol refers to entities with multiple components, such as vectors and matrices, the units given apply to each individual component of the entity. In the cases where the units are not listed, this is for one of several reasons. The symbol may not have any units associated with it, such as for strings and for a region of space (the location of the points within the space have units of length, but the set of points itself does not have an associated unit). Another reason for not including units is that the choice of units may be dependent on a particular instance of the symbol. For instance, the fluidity parameter (γ) is sometimes defined as $1/2\eta$, which will have units of $1/(\text{StressU}\cdot\text{t})$, but in other cases the fluidity parameter will have a different definition and different units. The potential ambiguity in units is a consequence of using a generic model of material behaviour.

Ω	- the region of space occupied by a body
x, y, z	- coordinates in Cartesian space (L) (m)
$\bar{\mathbf{T}}$	- surface traction (StressU) (Pa)
$\bar{\mathbf{u}}$	- vector of prescribed displacements (L) (m)
H_0, L_0, W_0	- the original height, length and width, respectively, of a rectangular brick (L) (m)
H, L, W	- the current height, length and width, respectively, of a rectangular brick (L) (m)
ΔL	- change in length between the original and the current configuration (L) (m)
A	- cross-sectional area (L^2)(m^2)
	- a material property for power-law and strain-hardening materials
σ	- true uniaxial stress (StressU) (Pa)
σ^E	- engineering uniaxial stress (StressU) (Pa)
ϵ	- true (natural) uniaxial strain (L/L) (m/m)
$\dot{\epsilon}$	- rate of change of true uniaxial strain (1/t) (1/s)
$d\epsilon$	- an infinitesimal increment in the natural uniaxial strain (L/L) (m/m)
ϵ^E	- engineering uniaxial strain (L/L) (m/m)
dL	- an infinitesimal increment in the length of a uniaxial member (L) (m)

E, E_1, E_2, E_3	- Young's modulus values (StressU) (Pa)
k	- spring stiffness (ForceU/L) (N/m)
	- parameter for Von Mises and Drucker-Prager constitutive equations (StressU) (Pa)
u	- displacement (L) (m)
η	- viscosity (StressU $\cdot$ t) (Pa $\cdot$ s)
	- parameter for Mohr Coulomb material
$\sigma_{y1}, \sigma_{y2}, \sigma_{y3}$	- the yield stress for different materials (StressU) (Pa)
$\lambda, \lambda_1, \lambda_2, \lambda_3$	- relaxation time values (t) (s)
α	- material property for Drucker Prager constitutive equation
c	- angle of cohesion (angleU) (rad)
ϕ	- friction angle (angleU) (rad)
ψ	- angle for Mohr-Coulomb (angleU) (rad)
$\boldsymbol{\sigma}$	- Cauchy (true) stress tensor (StressU) (Pa)
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	- normal stresses (StressU) (Pa)
$\sigma_{xy}, \sigma_{yz}, \sigma_{xz}$	- shear stresses (StressU) (Pa)
$\Delta\mathbf{f}$	- resultant force vector (ForceU) (N)
ΔS	- small element of a surface
S	- a surface
V	- a volume
P, Q, P_0, Q_0, P', P''	- points within a material continuum
$\hat{\mathbf{n}}$	- a unit outward normal vector
$\mathbf{t}^{(\hat{\mathbf{n}})}$	- traction on a surface oriented in the direction $\hat{\mathbf{n}}$ (StressU) (Pa)
$\dot{\boldsymbol{\epsilon}}$	- time rate of change of total natural strain (1/t) (1/s)
$d\boldsymbol{\epsilon}$	- increment in total natural strain (L/L) (m/m)
$\dot{\boldsymbol{\epsilon}}^e$	- time rate of change of elastic natural strain (1/t) (1/s)
$\dot{\boldsymbol{\epsilon}}^{vp}$	- time rate of change of viscoplastic natural strain (1/t) (1/s)
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	- normal strains (L/L) (m/m)
$\epsilon_{xy}, \epsilon_{yz}, \epsilon_{xz}$	- shear strains (L/L) (m/m)
$\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$	- shear strains as usually measured by engineers (L/L) (m/m)
$\mathbf{X}$	- material (Lagrangian) coordinate (L) (m)
X_1, X_2, X_3	- components of material (Lagrangian) coordinates in Cartesian space (L) (m)
$\mathbf{x}$	- spatial (Eulerian) coordinate (L) (m)
x_1, x_2, x_3	- components of spatial (Eulerian) coordinates in Cartesian space (L) (m)
$d\mathbf{X}$	- small change in the $\mathbf{X}$ vector (L) (m)
dX_1, dX_2, dX_3	- components of the small changes in the $\mathbf{X}$ vector (L) (m)
$d\mathbf{x}$	- small change in the $\mathbf{x}$ vector (L) (m)
dx_1, dx_2, dx_3	- components of the small changes in the $\mathbf{x}$ vector (L) (m)
$\mathbf{u}$	- displacement vector (L) (m)

u_1, u_2, u_3	- components of the displacement vector $\mathbf{u}$ (L) (m)
$d\mathbf{u}$	- small change in the $\mathbf{u}$ vector (L) (m)
du_1, du_2, du_3	- components of the change in the displacement vector $d\mathbf{u}$ (L) (m)
$\mathbf{F}$	- material deformation gradient (L/L) (m/m)
$\mathbf{I}$	- identity matrix
$\mathbf{J}$	- material displacement gradient (L/L) (m/m)
F	- a force (ForceU) (N)
	- the yield function
Q	- plastic potential function
κ	- hardening parameter
ϕ	- a function of F
γ	- fluidity parameter
ν	- Poisson's ratio
t, τ	- time (t) (s)
t_{begin}	- initial time (t) (s)
t_{end}	- final time (t) (s)
dt	- infinitesimal change in time (t) (s)
$\mathbf{D}$	- constitutive matrix (StressU) (Pa)
mat_prop_val	- material property values
mat_prop_names	- names of material properties
$name$	- name of material
$descript$	- description of the material
$\mathbf{s}_\sigma$	- summary of state of stress
$\mathbf{s}_\epsilon$	- summary of state of strain
σ_m	- first invariant of the stress tensor (StressU) (Pa)
p	- mean normal pressure (StressU) (Pa)
$\mathbf{s}$	- deviatoric stress tensor (StressU) (Pa)
$\mathbf{e}$	- deviatoric strain tensor (L/L) (m/m)
ϵ_v	- volumetric strain (L/L) (m/m)
J_2	- second invariant of the deviatoric stress tensor ($\mathrm{StressU}^2$) (Pa^2)
q	- effective stress (StressU) (Pa)
J_2^ϵ	- second invariant of the deviatoric strain tensor ($\mathrm{L}^2/\mathrm{L}^2$) ($\mathrm{m}^2/\mathrm{m}^2$)
ϵ_q	- effective strain (L/L) (m/m)
m	- power for power-law viscosity
μ	- parameter for Mohr Coulomb material
η^*	- parameter for Mohr Coulomb material
μ^*	- parameter for Mohr Coulomb material

Superscripts

$\dot{}$	- time derivative
E	- engineering measure
e	- elastic
vp	- viscoplastic

Subscripts

0	- original configuration or initial stress
$1, 2, 3$	- used to indicate different materials and used for indicating different coordinate axes

Prefixes

Δ	- finite change in following quantity
d	- infinitesimal change in the following quantity

Abbreviations and Acronyms

1D	- one dimensional
2D	- two dimensional
3D	- three dimensional
L	- Lagrangian
UL	- Updated Lagrangian
E	- Eulerian
CA	- Commonality Analysis
DSL	- Domain Specific Language

Types

$\mathbb{R}$	- real numbers
$\mathbb{R}^+ : \{x : \mathbb{R} \mid x \geq 0 : x\}$	- positive real numbers
poissonT : $\{x : \mathbb{R} \mid 0 < x \leq 0.5 : x\}$	- valid values for Poisson's ratio
vectorT : $\mathbb{R}^{3\times 1}$	- column vector
array1DT : $\mathbb{R}^{6\times 1}$	- one dimensional array representation of a symmetric tensor
array2DT : $\mathbb{R}^{6\times 6}$	- two dimensional array
state_zeroT : $\mathbb{B}^{6\times 1}$	- the components are true when a corresponding tensor value is known to be zero
angleT : $\{\theta : \mathbb{R} \mid 0 \leq \theta \leq 2\pi : \theta\}$	- valid angles (in radians)

1 Introduction

The modelling of deformation is necessary to solve many engineering problems, such as for determining the deflection of a structure, or the stresses in an airplane wing, or the thickness of a manufactured sheet of plastic film. For problems like these, where the material body cannot be assumed to be rigid, the conservation equations of mechanics (conservation of mass and conservation of momentum) do not provide enough information to solve for changes in the body's configuration. To determine deformation, another equation has to be introduced, the so-called closure or constitutive equation, which relates the deformation history of the body and the current stress field. A wide range of constitutive equations are used in engineering applications. For instance, materials may be modelled as elastic, viscous, viscoelastic, plastic or viscoplastic. Although the behaviour of these different types of materials can be very different, the mathematics used to describe them is similar. Using the correct abstraction it is possible to consider the above range of material behaviours as a family of material models.

To gain insight into this family of material models, this document presents a Commonality Analysis (CA) of the family. This CA for a family of material models follows the guidelines of a CA for a program family as found in Cuka and Weiss (1997) and Weiss and Lai (1999). The CA can be seen as a method for summarizing the requirements for all potential materials that are considered to be within the scope of the material model family. The CA includes documentation of terminology, commonalities (including goals, theoretical models and assumptions) and variabilities.

The template use for documenting this CA is based on the template for specialized physical problems presented in Lai (2004), Smith and Lai (2005) and Smith et al. (2007a). The structure of this document also borrows ideas from the template for general purpose scientific computing tools introduced in Smith (2006). In the remainder of this section the purpose of the report is described, the scope of the family is delineated and an outline of the remaining sections of the document is provided.

1.1 Purpose of Document

The purpose of this document is to summarize a family of material models, where each member model consists of equations, often called constitutive equations, that characterize the material's response to applied loads. The intention is that the models can be used to analyse continuum mechanics problems that arise within the context of engineering applications. In many cases the solution procedure will involve writing computer code and using algorithms from the field of computational mechanics. The CA documented here can be refined into part of a requirements document for a specific physical problem that requires a constitutive equation, or it can be used as the basis for a Domain Specific Language (DSL) (van Deursen et al., 2000). In the case of a DSL, a specification for a constitutive equation is written using the DSL and then the necessary code can be generated from this specification.

This document is intended to be a reference for those working with members of the family of materials documented herein. As such it is not intended that the document be read sequentially. Rather the reader will typically refer to the document on an as needed basis to consult the sections necessary to address their current concerns. To assist with this typical mode of usage, the structure of the document includes a table of contents, a table of units, a table of symbols, a table of types and a table of abbreviations. Further assistance with navigation of the CA is provided by explicitly documenting the relations between definitions, assumptions, goals, theoretical models and variabilities. In the case of the pdf version of the document, hyper-references are included to facilitate reading and searching of the document.

Although the theory, terminology and equations presented in this document are not new, the manner in which the information is presented is apparently unique. A reasonably complex theory is presented, but at the same time the document remains self-contained. To accomplish this goal and at the same time keep the size of the document small enough to be practical, no attempt is made to cover the breadth of continuum mechanics, but within the scope of the constitutive equations in the family, all necessary details are presented. An example of this approach is illustrated by the presentation of the definition of deformation that is given within this document. Although there are many potential measures of deformation, which are usually documented within continuum mechanics textbooks, the only one necessary in the current context, and thus the only one presented in this document, is the rate of natural strain tensor.

One way in which the current document adds value, over what could be achieved by simply reading the various sources on which it is based, is that it employs a consistent notation and terminology. Moreover, potential ambiguities that would exist when combining different documents are removed. For instance, in continuum mechanics many different measures of stress and strain are used, but a specific equation is only valid for the specific measures for which it was derived. It is not possible to simply swap one stress or strain measure for another, especially in the case of large deformations. To remove this potential ambiguity, the stress and strain measures associated with the presented equations are clearly defined. Given that irrelevant details are left out of the documentation, one may be concerned that implicit assumptions about relevance were made during document preparation. To guard against this problem, an effort is made to clearly document all assumptions. Moreover, the documented assumptions provide the important role of assisting with delimiting the scope of the family.

1.2 Scope of the Family

The scope of this family of material models was chosen to keep the family simple, but at the same time allow for the use of the family members in many different contexts. For this reason the scope of this document is only constitutive equations; the other equations necessary to solve a continuum mechanics problem have been excluded. By excluding other equations the amount of detail required is reduced, which facilitates reusing the constitutive equation in a variety of situations. The particulars choices with respect to the scope of the family are as follows:

1. The focus of the mathematics is on a single material particle and the region immediately around it. The concept of a mathematical limit is employed so that the equations describe stress at the point in space that the material particle currently occupies. To describe the stress field within a body the stress at each point within the body has to be considered.

2. The equations described here cannot be used on their own to solve computational mechanics problems. The equations describe the dependence of stress on the history of strain, but the stress cannot usually be determined from these equations alone, as the strain history is not usually explicitly given. Rather than explicitly knowing the strain history, the analyst will solve for stress and strain by combining the constitutive equation with the kinematic equation defining strain with respect to displacements, the equilibrium equation, and the problem boundary conditions and initial conditions. That is, the constitutive equations will in general be coupled with other equations and determination of the stress will involve solving a larger system of equations than those presented in this CA.

3. The constitutive equations within this family will all be of the Perzyna type (Perzyna, 1966). Equations of this kind can describe materials that are elastic, viscous, plastic, viscoelastic or viscoplastic, so a wide range of potential behaviours are covered. However, the restriction to one class of constitutive equations means that other classes of constitutive equations are excluded, such as the following: hyperelastic (Malvern, 1969, pages 282–288), hypoelastic (Mase, 1970, page 149), differential constitutive equations (Joseph, 1990), and integral constitutive equations (Joseph, 1990).

4. The constitutive equations are written in the rate form; that is, the equations involve time derivatives of stress and strain. In some of the literature on constitutive equations the same equations are presented in incremental form, since the equations are nearly always numerically solved for discrete time steps. The rate form was adopted because it is more elegant mathematically and it includes all of the same information as the incremental form. In fact, the incremental form can be obtained from the rate form simply by multiplying by the size of a time step (Δt).

5. The vectors and tensors presented in this document employ a rectangular Cartesian coordinate system; that is, the basis vectors consist of a set of orthonormal vectors. This decision was made to simplify the presentation. A more general family would allow for different coordinate systems, such as cylindrical and spherical coordinate systems, but at the expense of additional abstraction. In all engineering contexts the Cartesian system can be made to work, and in most contexts it is the natural and simplest choice. Therefore, a simpler, less abstract, notation is adopted, to make this document relevant to as wide an audience as possible.

6. Nonfunctional requirements are not covered in the CA. Although the desired software qualities, such as accuracy, efficiency, portability etc., will certainly

be variabilities between family members, the connection between the requirement and the design cannot be easily quantified or predicted. Therefore, the specification of nonfunctional requirements is left as program specific, rather than as an explicitly identified variability of the family of material models.

7. Computational mechanics algorithms are outside of the scope of this work. The CA is a requirements document for a family; as such, its focus is on "what" equations need to be solved, not on "how" to solve them. Furthermore, in this initial stage of development it is premature to determine a specific algorithm, since the choice will depend on equations other than the constitutive equation. The choice of the appropriate numerical algorithm is left to the design stage.

1.3 Organization of the Document

As mentioned previously this document follows a structure adapted from Lai (2004), Smith and Lai (2005), Smith (2006) and Smith et al. (2007a). The template that has been adopted most closely follows that in Smith (2006). However, a section explicitly describing the scope of the family (Section 1.2) has been added. Also, the assumptions section from Smith (2006) has been moved from being a variability to being a commonality. This change was made because Smith (2006) describes general purpose scientific computing tools, which typically are distinguished by the assumptions that the tool is allowed to make. In the current context the document is describing a physical model that is common to all family members. Therefore, the simplifying assumptions that were made in the model's derivation are commonalities. This is the same approach used in Lai (2004), Smith and Lai (2005) and Smith et al. (2007a) for documenting specific physical problems.

The structure of this document is essentially top down, with details added as one proceeds through it. This first section of the CA serves the purpose of introducing the family of material models, while the second section provides a general description of the physical problem and the different contexts where a material model might be used. This second section includes subsections listing potential system contexts, user characteristics and system constraints. The third section presents the terminology and requirements that are common to all members of the family. This includes assumptions, the goal statement and the theoretical model adopted. The fourth section describes the variabilities that distinguish the family members. The fifth section includes dependence graphs that show the relationship that exists between data definitions, goal statements, assumptions, theoretical models and the variabilities. The final section provides examples of potential family members, by explicitly stating the values of the different variabilities for several different classes of material.

2 General System Description

Constitutive equations are important in the fields of continuum and computational mechanics. In continuum and computational mechanics the goal is often to solve for the changes over time of deformation and stress within some given body Ω, as

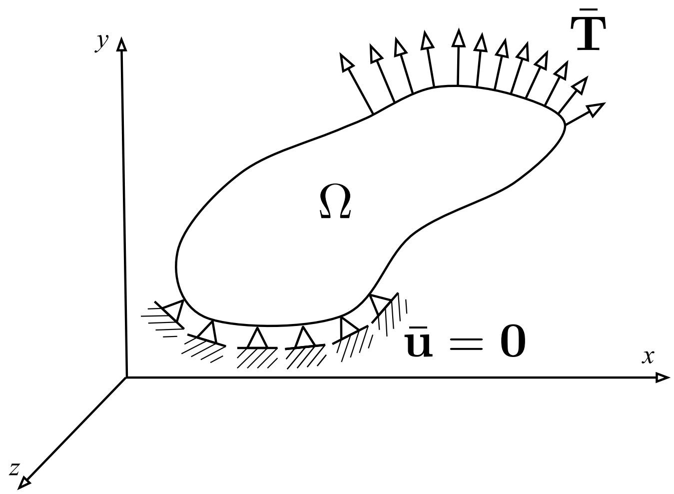

Figure 1: A typical boundary valued problem with prescribed displacements and an applied surface traction

depicted in Cartesian (x-y-z) space in Figure 1. To solve for the histories of displacement, strain and stress it is necessary to satisfy the governing partial differential equation for equilibrium, subject to initial conditions and boundary conditions. The initial conditions involve setting initial stress and strain fields, while the boundary conditions involve specifying the surface traction $\bar{\mathbf{T}}$, or the prescribed displacements $\bar{\mathbf{u}}$.

However, the equilibrium condition alone does not provide enough equations to solve for all of the unknowns. In three dimensions the equilibrium condition provides three (3) equations, one to state the balance of force in each of the three coordinate directions. However, in a general three dimensional problem, there are 15 unknowns, comprised of 6 unknown stress components, 6 unknown strain components and 3 unknown displacements. To obtain the necessary additional 12 equations, the 6 kinematic equations that relate strain and displacement are introduced along with 6 constitutive equations. The constitutive equations postulate a dependence of the stress on the history of deformation. The constitutive equation then provides the necessary material specific information that allows the analyst to determine the history of deformation and stress within the body over the time of interest.

The input to a computational mechanics program consists of initial conditions for stress and strain; boundary conditions for tractions and displacements (possibly changing over time); numerical parameters, such as error tolerances; and material properties. The last input of material properties is directly tied to the constitutive equations, as these equations provide the information that is specific to a given material. Material properties provide measures of elasticity, viscosity, relaxation time, density, cohesion, etc. Most of the necessary material specific information is determined by the needs of the constitutive equations, although the property of density may instead enter the problem as part of the equilibrium equation, if the body force of self-weight is included in the formulation. The output of the computational mechanics program consists of deformation and stress histories. In many formulations the output is discretized over space and time. That is, the stress values and deformation measures are given at discrete locations in space and separate points in time.

The sections that follow list potential system contexts, user characteristics and system constraints. The CA covers all the potential members of the material model family, so it is not possible to know exactly the uses to which the document will be applied in the future; however, information is recorded on typical uses of the members of the material model family. Although the information in this section cannot be presented as variabilities, because it does not represent requirements, the hints provided in the CA can later be refined when a material model is used in a specific continuum or computational mechanics problem.

2.1 Potential System Context

The individual family members will be used in the context of analysis problems in continuum mechanics. For a given material, geometric configuration, initial conditions and boundary conditions, the analyst's job is to determine the history of deformation of the body and the resulting stress field. This document's focus on analysis, rather than design, is not a significant limitation because the design process can often be viewed as a series of analysis problems. In the case of design, the material properties, and potentially even the material model itself, are unknowns that need to be determined so that the solution is optimal in some respect. For instance, the design may optimize for a high strength to weight ratio. In the search for this optimal material many analysis problems will be solved and their solutions compared.

When the time arrives to actually solve the analysis problem, many approaches are available. For the simplest problems a "by hand" calculation may suffice, but in many cases a numerical solution will be sought. When numerical algorithms are employed the context changes from continuum mechanics to computational mechanics. Some methods that are employed in computational mechanics include the finite difference method (FDM), the finite volume method (FVM), the finite element method (FEM) (Zienkiewicz et al., 2005), the method of lines, spectral methods, etc.

The family of material models documented in this CA is best suited to problems in solid mechanics, as opposed to fluid mechanics. Although the constitutive equation can approximate the viscous behaviour that many fluids possess, the theory is based on a material particle, rather than a region of space, so it is better suited to solid mechanics problems. More concretely, the constitutive equation documented here views the material particle as an independent variable and describes the particle's history of deformation and stress. If instead the independent variable were a location in space and the constitutive equation were written in terms of the material particle instantaneously at that location, then problems in fluid mechanics could be handled in a more natural manner.

Within the field of solid mechanics there are many potential contexts in which a material model may appear. For instance, members of the family of material discussed here have been used to model polymers (Smith and Stolle, 2003), soils (Borja and Lee, 1991), concrete and metals.

Rather than using a single material model, the context may instead consider multiple members of the family. In this case a family member may be specified through the use of a DSL. Given a DSL specification the appropriate code to represent the model can be generated. An approach like this could potentially be used in developing a virtual laboratory for material testing (McCutchan, 2007; Smith et al., 2007b; Gao, 2004; Smith and Gao, 2005).

2.2 Potential User Characteristics

All users of a member of the material model family should at least have the following knowledge or equivalent:

1. Physics for first year university science or engineering students
2. Calculus for first year university science or engineering students
3. Linear algebra for first year university science or engineering students

Additional user characteristics depend on the context in which the material model appears. If the problem is to be solved using a computer, then the user must have an ability to interact with computers, specify input and process the results. If the user is going to solve the problems by hand, then a graduate level understanding of continuum mechanics is necessary. If the user will be writing a computer program, then they will also need a graduate level of understanding of computational mechanics and at least one course on programming. If the user is simply using a program that someone else has written, then a Level II university course on engineering mechanics should be adequate.

2.3 Potential System Constraints

There does not appear to be a need to place constraints on the implementation of the material model. This is fortunate as the presence of constraints is undesirable in a requirements document because constraints impose design solutions too early in the development process. One conceivable exception to the absence of constraints is when the material model will be incorporated into an existing computational mechanics program. In this case, it may be reasonable to place a constraint on the programming language used for implementation.

3 Commonalities

This section lists all the common requirements among all of the potential family members. The first subsection below provides background information that is common to all material models. This information includes an overview of the concepts of stress and strain and examples of various material behaviours including elastic, viscous, plastic and viscoplastic. The background examples are for a one-dimensional (1D) state of stress, as this is simple to present and understand. In the subsequent subsection on terminology definitions the concepts of stress and strain are generalized to be three-dimensional (3D). Besides definitions for stress and strain the terminology section provides other data definitions, such as the yield function and the plastic potential function. The data definitions are used for the later presentation of the assumptions and of the theoretical model. The subsection on terminology is followed by a subsection that consists of the goal statement that is common to all members of the material model family. After this, assumptions are listed, where the assumptions allow the goal statement to be refined into the theoretical model given in the last subsection.

3.1 Background Overview

Material models provide equations that define the relationship between the stress (load) and the strain (deformation). These models are a mathematical approximation of real world material behaviour. Many different models for material behaviour exist. The family of models discussed here includes elastic, viscous and plastic material behaviour. As well as combinations of these three behaviours. In this section the differences between elastic, viscous and plastic material behaviours will be presented along with common models of each. The models will be presented using a common experiment, the uniaxial extension of a rod, as shown in Figure 2. The test specimen is a rectangular box with the original dimensions of $L_0 \times W_0 \times H_0$. A force F is applied to the free end of the specimen so that it deforms to the new dimensions of $L \times W \times H$. This experiment is 1D, which allows illustration of the important points, without the need to introduce unnecessary details. Before describing the three models of material behaviour and illustrating the typical usage of constitutive equations, a brief introduction to the definition of stress and strain is presented.

3.1.1 Stress and Strain

Considering a uniaxial extension experiment, as shown in Figure 2, stress is defined as the force (F) divided by the current (deformed) cross-sectional area ($A = WH$). This stress can be denoted by σ.

$$\sigma = \frac{F}{A} \tag{1}$$

A distinction can be made between true stress and engineering stress. The above equation is for true stress, which references the deformed configuration. The engineering stress (σ^E), on the other hand, references the original undeformed configuration, as follows:

$$\sigma^E = \frac{F}{A_0} \tag{2}$$

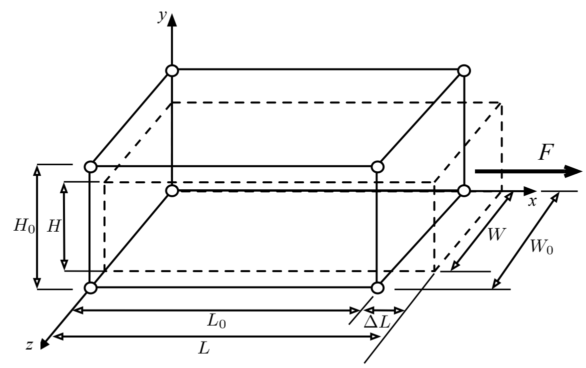

Figure 2: Test specimen undergoing uniaxial extension test

where $A_0 = W_0 H_0$ is the original cross-sectional area of the rod. The true stress definition takes into account that deformations will occur under loading and thus change the area that the force is applied to. In a typical uniaxial extension experiment deformation will lead to a significant decrease in the cross-sectional area of the member; this phenomenon is known as "neck-in."

Strain, which is used as a dimensionless measure of deformation, is denoted by ϵ. It is easier to define engineering strain before true strain, because of the latter's relative complexity. Engineering strain, denoted here by ϵ^E is defined as the change in the length (ΔL) of the rod over the original length (L_0).

$$\epsilon^E = \frac{\Delta L}{L_0} \tag{3}$$

Unlike engineering strain, true strain takes into account the history of length changes. The true strain is defined by first considering a small strain increment ($d\epsilon$), which is defined as follows:

$$d\epsilon = \frac{dL}{L} \tag{4}$$

where dL is the current increment in the length and L is the current rod length. By summing all strain increments over the course of a given deformation the true strain ϵ is defined by the following equation:

$$\epsilon = \int_{L_0}^{L_0+\Delta L} \frac{dL}{L} = \ln\left(\frac{L_0 + \Delta L}{L_0}\right) = \ln\left(\frac{L}{L_0}\right) \tag{5}$$

where L in the last equation is the final length of the test specimen.

For very small deformations, which are the most common in practice, the engineering stress and strain are essentially equivalent to the true values and thus remain physically meaningful and useful for most engineering purposes. For a more

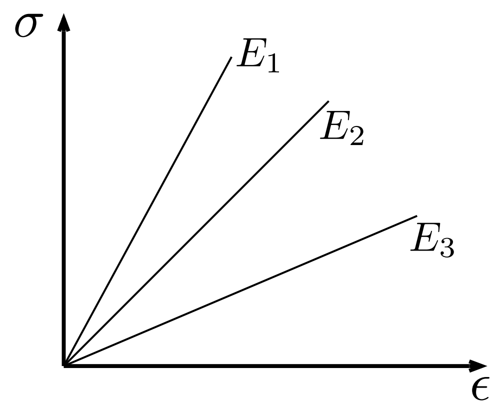

Figure 3: Typical elastic stress versus strain graphs $E_1 > E_2 > E_3$

detailed discussion on stress and strain see Beer and Johnston (1985). In later sections the true stress and true strain are also referred to as the Cauchy stress and the natural strain, respectively.

3.1.2 Elasticity

Elastic materials are materials that deform when loaded and return to their original configuration after the load is removed. An elastic materials can be modelled as a spring that follow Hooke's law. The stress in an elastic material is linearly related to the strain of the material, as follows:

$$\sigma = E\epsilon \tag{6}$$

where E is known as Young's modulus or the elastic modulus. It is easy to see the connection with the spring equation $F = ku$, where k is the spring stiffness and u is the displacement of the spring. An elastic model is generally only accurate for small strains (say less than 10 %). A graph of stress versus strain for an elastic material with three different values of E can be seen in Figure 3.

3.1.3 Viscosity

Viscosity describes the stress that develops in a material to resist a given rate of deformation. Viscosity is typically associated with fluids. A material with high viscosity such as honey, resists a higher rate of deformation than a material with a low viscosity such as water. The stress of a viscous material depends on the true (natural) strain rate $\dot{\epsilon}$ and the coefficient of viscosity η as follows:

$$\sigma = 2\eta\dot{\epsilon} \tag{7}$$

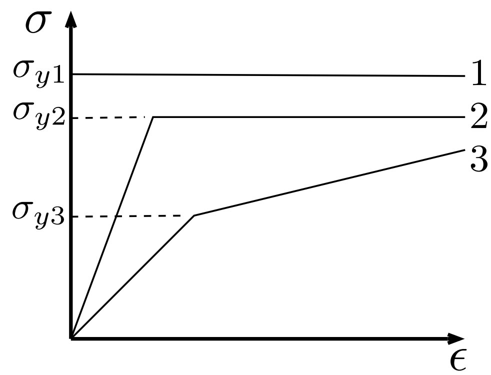

Figure 4: Typical plastic stress versus strain graphs showing: (1) perfectly plastic, (2) elastic perfectly plastic and (3) strain hardening

3.1.4 Plasticity

Unlike elastic deformation, plastic deformation is permanent. An elastoplastic material combines elastic and plastic behaviour. These materials begin deforming elastically until the material yields, after which subsequent deformation is permanent. Materials that can undergo large plastic deformations without fracturing, such as steel, are described as ductile. In contrast, materials that fracture suddenly, such as concrete, are called brittle materials. Some example plots of stress versus strain for plastic materials can be seen in Figure 4. Plastic deformation begins once the stress has reached the yield point (σ_y); this can be seen in Figure 4 for three materials with yield stresses of σ_{y1}, σ_{y2} and σ_{y3}, respectively. Perfectly plastic materials do not deform until the yield stress is met and elastoplastic materials first exhibit elastic behaviour and then plastic behaviour after yielding. Strain hardening is the phenomenon of the yield stress getting larger as the material undergoes additional straining past the initial yield limit.

3.1.5 Viscoplasticity

Materials can have behaviour that is a combination of elastic, viscous and plastic material behaviours. These types of materials are labelled viscoplastic materials. Example viscoplastic materials include metals, soils, and molten polymers. The viscoplastic material can be modelled as a purely elastic spring in series with a viscoplastic dashpot. A plot of stress versus strain for a viscoplastic material under a constant rate of natural strain can be seen in Figure 5. The plot includes three different relaxation times. Relaxation time is a measure of how quickly the elastic stress relaxes. Low values of λ, such as λ_1 in Figure 5, approximately correspond to the viscous behaviour discussed in Section 3.1.3.

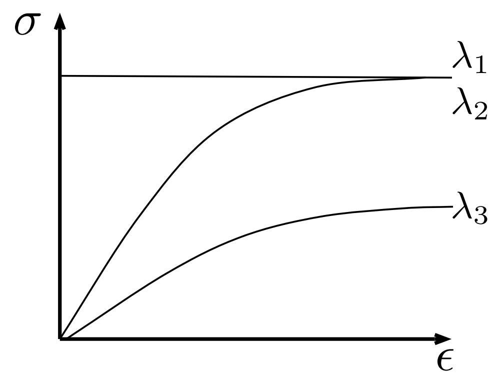

Figure 5: Typical viscoplastic stress vs. strain graphs where the relative relaxation times are ordered $\lambda_1 < \lambda_2 < \lambda_3$ (constant strain rate test)

3.1.6 Solving for Stress, Strain and Deformation

This document defines the constitutive equation in isolation, but the constitutive equation is rarely used on its own. Instead, the constitutive equation is used together with other equations to solve continuum mechanics problems where a scientist or engineer desires to find the deformation and stress within a body under a prescribed load or displacement. For the one dimensional uniaxial extension example of this section, there are three unknown functions: stress $\sigma(x)$, strain $\epsilon(x)$ and displacement $u(x)$, where the displacement gives the distance moved by the particle at x as it deforms from the initial to the final configuration.

For this 1D problem, we have three unknowns and thus require three equations. The first equation comes from the need to satisfy the equilibrium principle, which says that the forces in the x-direction should be balanced. This means that the internal force (σA) should match the external force F. Stated another way,

$$\frac{d(\sigma A)}{dx} = 0, \text{ with } \sigma(L)A = F, \text{ for a constant } A \text{ this implies } \frac{d\sigma}{dx} = 0 \tag{8}$$

The above equation allows for the solution $\sigma(x) = F/A$, but it does not provide information on the deformation. To find this, we need to introduce a constitutive equation. For illustration purposes, the equation for a linear elastic solid is reproduced here:

$$\sigma = E\epsilon \tag{9}$$

The constitutive equation provides the strain, $\epsilon = \sigma/E = F/AE$. To get the final displacement of the body, the kinemenatic equation relating strain and displacement can be introduced, as follows:

$$\epsilon = \frac{du}{dx} \tag{10}$$

A unique solution to the above equations requires that the appropriate boundary conditions be set. The equilibrium condition as written above already mentions the load boundary condition on the right hand side of the uniaxial member. To keep the member from moving, a displacement constraint must also be added for the left hand side, as follows:

$$u(0) = 0 \tag{11}$$

Using this boundary condition, $u(x) = \frac{F}{AE}x$

In the general 3D case, as discussed in Section 2, we have 15 unknowns and require 15 equations, but the source of these three equations is the same three principles outlined above: equilibrium equations, constitutive equations and kinematic equations. Moreover, the general 3D case, as in the 1D example, also requires boundary conditions for loading and displacement to yield a unique solution.

3.2 Terminology Definition

Throughout this document consistent units are employed and a consistent notation is used. As described in the preamble to the Table of Symbols, the unit system adopted is the "MLtT" dimension system, where M is the dimension of mass, L is length, t is time and T is temperature. This system corresponds nicely with the SI (Système International d'Unités), or modern metric, system, which uses units of kilogram (kg), meter (m), second (s) and Kelvin (K) for M, L, t and T, respectively. By leaving the units in this form any unit system can be adopted in the future, as long as the choice of specific units is consistent between different quantities. When units are listed in this report for matrix and vector quantities, it is implicit that the unit measure applies to the individual components of the matrix or vector. Furthermore, when units are listed, the generic (MLtT) units will be given first, followed by the equivalent SI units. Generally the units will be placed within parentheses.

With respect to mathematical notation, this document employs conventional vector and matrix notation, with scalars in a normal font and both vectors and matrices represented by a bold face font. A dot over a symbol means differentiation with respect to time. The convention followed is the same as that used in Kreyszig (1988, pages 307–580). Alternative notations that could have been used include symbolic (Gibbs) notation (Mase, 1970, pages 2–8) and Einstein's index notation (Mase, 1970, 8–11), (Einsten, 1916). As the discussion of stress and strain will illustrate, the convention will be adopted that two dimensional symmetric tensors can be represented by 1D arrays, as in Zienkiewicz et al. (2005).

This section consists of definitions of terminology that are necessary to understand the modelling of material behaviour. These definitions will be used later to explain the assumptions, goal statement, theoretical model and variabilities. Although in some cases the definitions will look like some of the variables that are introduced later, this section is describing concepts, not specific variables. For instance, it is the concepts of stress that is presented here, not a variable; the concept of stress will later be used to understand the input variable of initial stress and the output variable of stress history. The ordering of the definitions was selected so that definitions that depend on information from other definitions appear in the list after these other definitions.

Each definition in this section uses the same table structure, with the following rows:

Number: All of the data definitions are assigned a unique number, which takes the form of a natural number with the prefix "D." This number will be used for purposes of cross-referencing and traceability within this document.

Label: The label is a short identifying phrase, each with the prefix "D_." This label provides a mnemonic that helps with quickly remembering which definition is being presented. Moreover, the label will useful when an external document needs to reference one of the definitions in this document.

Symbol: This field shows the symbol that is used to represent variables related to this concept. For instance, the natural strain rate tensor is represented by the symbol $\dot{\boldsymbol{\epsilon}}$. Later in the document when the symbol $\dot{\boldsymbol{\epsilon}}$ appears, possibly with superscripts such as e for elastic or vp for viscoplastic, this is an indication that the term in question is a measure of natural strain rate. If the symbol should appear without the dot over it, then it is referring to a total strain and not the rate of strain.

Type: Each variable designated by a symbol has a type associated with it, which is listed in this field of the data definition template. The type information helps to clarify the meaning of the symbol and the variables.

Units: Where applicable the units associated with the symbol are given. These units are given in terms of the mass (M), length (L), time (t) and temperature (T). For convenience the units are also given in SI.

Related Items: A related item is a data definition or assumption that is used by the current definition. That is, if the used data definition or assumption should change, then the current data definition will also need to be modified. As mentioned above, data definitions have a prefix "D." Assumptions will have the prefix "A."

Sources: This field lists references that can be consulted for additional information on the concept in question.

Description: The actual definition is given here. In some cases where the description is lengthy, some of the details are moved to a section following the table. When appropriate the description will reference the related definitions and assumptions.

History: Each data definition ends with a history of the definition, including the creation date and any subsequent modifications.

Number:	D1
Label:	D_Stress
Symbol:	$\boldsymbol{\sigma}$
Type:	array1DT
Units:	Each component of the stress tensor has units of StressU (Pa)
Related Items:	A1, A4, A6, A12
Sources:	Long (1961, pages 35–41); Malvern (1969, pages 64–119); Mase (1970, pages 44–76); Beer and Johnston (1985, pages 1–21)
Description:	The stress provides a measure of force per unit area associated with different directions at a point within a body. A detailed definition of the stress tensor is provided below. This definition is for the true stress, which is also sometimes called the Cauchy stress.
History:	Created – June 14, 2007

Detailed Description of Stress

To define the (true) stress $\boldsymbol{\sigma}$ at a point P within a material continuum that occupies the region of space Ω, one first needs to define the traction $\mathbf{t}^{(\hat{\mathbf{n}})}$ (also known as the stress vector) at the point P associated with direction $\hat{\mathbf{n}}$. The traction is defined by considering the interaction of an arbitrary volume V surrounding point P with the material outside this volume. The interaction will be across the surface S that encloses V. The material outside of V will exert a resultant force $\Delta\mathbf{f}$ across a small element ΔS of S. The unit outward normal from ΔS is represented by $\hat{\mathbf{n}}$. The average force per unit area on ΔS is given by $\Delta\mathbf{f}/\Delta S$. Figure 6 shows a sketch of the resultant force acting on the small element of the surface ΔS. The coordinate system assumed here is the rectangular Cartesian coordinate system (A6).

The Cauchy Stress Principle asserts that as ΔS approaches zero, $\Delta\mathbf{f}/\Delta S$ approaches the definite limit $d\mathbf{f}/dS$, while at the same time the moment of $\Delta\mathbf{f}$ vanishes (Mase, 1970). This limiting process is only possible if the continuum hypothesis (A1) is assumed to apply. Using the Cauchy Stress Principle the traction at point P associated with a particular surface element ΔS, which has unit outward normal $\hat{\mathbf{n}}$, is as follows:

$$\mathbf{t}^{(\hat{\mathbf{n}})} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{f}}{\Delta S} = \frac{d\mathbf{f}}{dS} \tag{12}$$

A different traction is associated with each different choice of $\hat{\mathbf{n}}$. However, it is possible to summarize the state of stress $\boldsymbol{\sigma}$ at a point by only considering three choices of $\hat{\mathbf{n}}$. If the traction is found for three mutually perpendicular planes at P, then the traction for any other plane through P can be found using coordinate transformation equations. Therefore, the state of stress at point P can be represented by three tractions acting on three faces of a cube. These tractions are shown

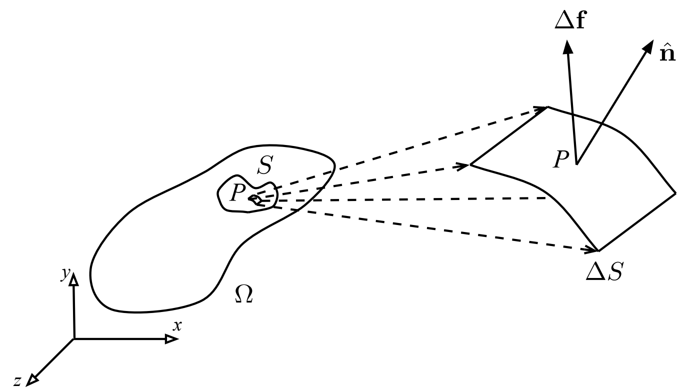

Figure 6: Resultant force ($\Delta\mathbf{f}$) on a small element (ΔS) of the surface (S) enclosing an arbitrary volume (V) about a point (P) within the material continuum occupying a region of space (Ω)

in component form in Figure 7.

In Figure 7 the components perpendicular to the planes (σ_{xx}, σ_{yy} and σ_{zz}) are termed normal stresses. The components acting tangent to the planes (σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx} and σ_{zy}) are known as shear stresses. A stress component is defined to be positive when it acts in the positive direction of the coordinate axes and on a plane whose outer normal points in one of the positive coordinate directions. This means that all of the components shown in Figure 7 are positive. The nine stress components form the stress tensor, which in matrix form is represented as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{13}$$

Moment equilibrium for any arbitrary volume of a continuum, subjected to surface tractions and body forces and assuming that there are no distributed moments or couple stresses (A4), implies that the stress tensor is symmetric (Mase, 1970, pages 48–49); that is, the following relationships hold:

$$\sigma_{xy} = \sigma_{yx}, \sigma_{yz} = \sigma_{zy}, \sigma_{xz} = \sigma_{zx} \tag{14}$$

Given these relationships, the stress tensor has 6 independent stress components, as opposed to 9. This allows for the convenient notation, as adopted in this report, of representing symmetric second order tensors as one dimensional arrays of dimension 6×1. Using the array notation, which is adopted for the remainder of this document, the stress tensor is written as follows:

$$\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{xz} \end{bmatrix} \tag{15}$$

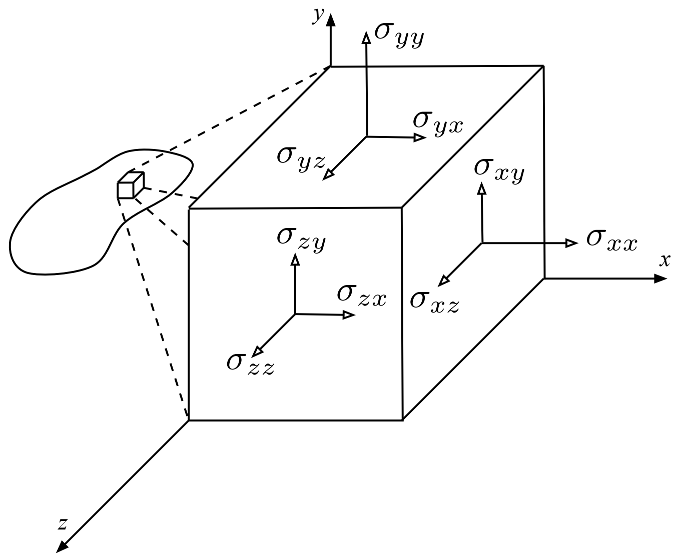

Figure 7: Stress tensor for a point within a body

The above definition of stress is termed the true stress or the Cauchy stress because it focuses on the current (deformed) configuration. The Cauchy stress is the natural stress measure to choose when the description of motion focuses on the material as it moves over time (Assumption A12). If the stress measure instead references the original configuration, then one has the engineering stress, which in continuum mechanics is called the Piola-Kirchoff stress (Malvern, 1969, pages 20–224).

Number:	D2
Label:	D_StrainRate
Symbol:	$\dot{\boldsymbol{\epsilon}}$
Type:	array1DT
Units:	Each entry in the strain rate tensor has units of 1/t (1/s).
Related Items:	A1, A5, A6, A12
Sources:	Long (1961, pages 51–58); Malvern (1969, pages 120–196); Mase (1970, pages 77–109); Beer and Johnston (1985, pages 31–38)
Description:	The natural strain rate tensor is a kinematic measure that represents the rate of deformation for a point within a body. A detailed definition of the natural strain rate tensor is provided below.
History:	Created – June 15, 2007

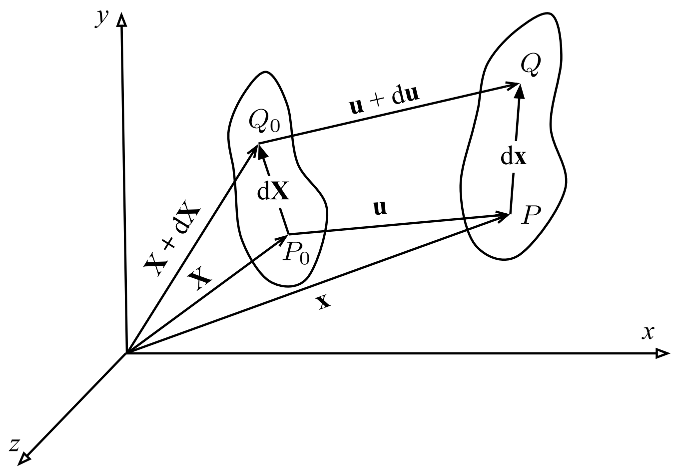

Figure 8: The initial configuration of neighbouring points P_0 and Q_0 moving to their deformed configuration P and Q, respectively

Detailed Description of Natural Strain Rate Tensor

The deformation of neighbouring material particles within a continuum is shown in Figure 8. The coordinate system assumed here is the rectangular Cartesian coordinate system (A6). To assist in the later derivations, some notation and terminology will now be introduced. In continuum mechanics $\mathbf{X}$, $\mathbf{x}$ and $\mathbf{u}$ (as shown in Figure 8) are usually called the material coordinates, spatial coordinates and the displacement vector, respectively. The vector $d\mathbf{X}$ connects neighbouring material particles (at points P_0 and Q_0) at time t and the vector $d\mathbf{x}$ connects these same two particles (now at points P and Q) at time $t + dt$, where dt is an infinitesimal quantity. In component form the vectors introduced above can be represented as follows:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} d\mathbf{X} = \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} d\mathbf{x} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{16}$$

In Figure 8 the configuration at time t can be considered as the reference configuration, or the undeformed configuration. To measure the deformation over the differential time step dt one can consider the change in the distance between the neighbouring particles P_0 and Q_0 to P and Q; that is, the change in length of $d\mathbf{X}$ to $d\mathbf{x}$. A common measure used to characterize the deformation is the change in the square of the length of the distances between the neighbouring particles as follows:

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{x}^T d\mathbf{x} - d\mathbf{X}^T d\mathbf{X} \tag{17}$$

This measure of deformation is appropriate because it excludes rigid body rotation. Insight into Equation 17 can be gained by finding a relationship between $d\mathbf{x}$ and $d\mathbf{X}$. Such a relation exists because the spatial coordinates $\mathbf{x}$ are a function of the material coordinates $\mathbf{X}$; that is $\mathbf{x} = \mathbf{x}(X_1, X_2, X_3)$. This relationship, together with the continuum hypothesis A1, allows one to write the total differential. For instance, the total differential for dx_1 can be written as:

$$dx_1 = \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 + \frac{\partial x_1}{\partial X_3} dX_3 \tag{18}$$

The total differential can be found for all components of $d\mathbf{x}$ and summarized as follows:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \quad F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad i, j \in \{1, 2, 3\} \tag{19}$$

where $\mathbf{F}$ is known as the material deformation gradient.

Equation 19 can be substituted into Equation 17 as follows:

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X}^T \mathbf{F}^T \mathbf{F} d\mathbf{X} - d\mathbf{X}^T d\mathbf{X} = d\mathbf{X}^T (\mathbf{F}^T \mathbf{F} - \mathbf{I}) d\mathbf{X} \tag{20}$$

where $\mathbf{I}$ is the identity matrix.

Up to this point $\mathbf{F}$ is expressed in terms of spatial and material coordinates, but it is customary to think about the deformation in terms of the displacement $\mathbf{u}$. From Figure 8, one can see that $\mathbf{x} = \mathbf{X} + \mathbf{u}$. The material deformation gradient can then be expressed as follows:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial X_i}{\partial X_i} + \frac{\partial u_i}{\partial X_j} \quad \text{or} \quad \mathbf{F} = \mathbf{I} + \mathbf{J}, \text{where } J_{ij} = \frac{\partial u_i}{\partial X_j} \tag{21}$$

where $\mathbf{J}$ is known as the material displacement gradient.

Equation 21 can be substituted into Equation 20 as follows:

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X}^T ((\mathbf{I} + \mathbf{J}^T)(\mathbf{I} + \mathbf{J}) - \mathbf{I}) d\mathbf{X} = d\mathbf{X}^T (\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T \mathbf{J}) d\mathbf{X} \tag{22}$$

If one assumes that the material displacement gradients are small, as given in Assumption A5, then the higher order term $\mathbf{J}^T \mathbf{J}$ can be assumed to be negligible, which means that

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X}^T (\mathbf{J} + \mathbf{J}^T) d\mathbf{X} = d\mathbf{X}^T (2 d\boldsymbol{\epsilon}) d\mathbf{X} \tag{23}$$

where $d\boldsymbol{\epsilon}$ is an increment of the natural strain tensor of the time step dt.

$$d\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T) \quad \text{or} \quad d\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{24}$$

Equation 24 can be divided by the time increment dt to obtain the rate form of the natural strain.

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}(\dot{\mathbf{J}} + \dot{\mathbf{J}}^T) \quad \text{or} \quad \dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right) \tag{25}$$

where v_i ($\mathbf{v}$) is the velocity vector.

The nine strain rate components form the natural strain rate tensor, which in matrix form is represented as follows:

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} \dot{\epsilon}_{xx} & \dot{\epsilon}_{xy} & \dot{\epsilon}_{xz} \\ \dot{\epsilon}_{yx} & \dot{\epsilon}_{yy} & \dot{\epsilon}_{yz} \\ \dot{\epsilon}_{zx} & \dot{\epsilon}_{zy} & \dot{\epsilon}_{zz} \end{bmatrix} \tag{26}$$

The definition of the natural strain rate tensor (Equation 25) shows that this is a symmetric tensor. Therefore, just as for the stress tensor, it is convenient to write the 6 independent components using array notation, as follows:

$$\dot{\boldsymbol{\epsilon}}^T = \begin{bmatrix} \dot{\epsilon}_{xx} & \dot{\epsilon}_{yy} & \dot{\epsilon}_{zz} & \dot{\gamma}_{xy} & \dot{\gamma}_{yz} & \dot{\gamma}_{xz} \end{bmatrix} \tag{27}$$

where $\dot{\gamma}_{xy} = 2\dot{\epsilon}_{xy}$, $\dot{\gamma}_{yz} = 2\dot{\epsilon}_{yz}$ and $\dot{\gamma}_{xz} = 2\dot{\epsilon}_{xz}$. The introduction of γ in place of ϵ is a convention often used in computational mechanics. The first three components in the natural strain rate array (Equation 27) correspond to normal strains and the last three correspond to shear strains. As for the stress tensor, the strain tensor will be represented as a 1D array for the remainder of this document.

The natural strain rate tensor described in this definition corresponds to the true strain discussed in Section 3.1.1 because the deformation is always relative to the previous state. This previous state will generally be a deformed configuration relative to its previous state and so on until one returns to the initial undeformed configuration. In terms of the descriptions of motion given in Description D6, the natural strain rate comes from the UL (Update Lagrangian) description of motion, with Δt infinitesimally small, as stated in Assumption A12.

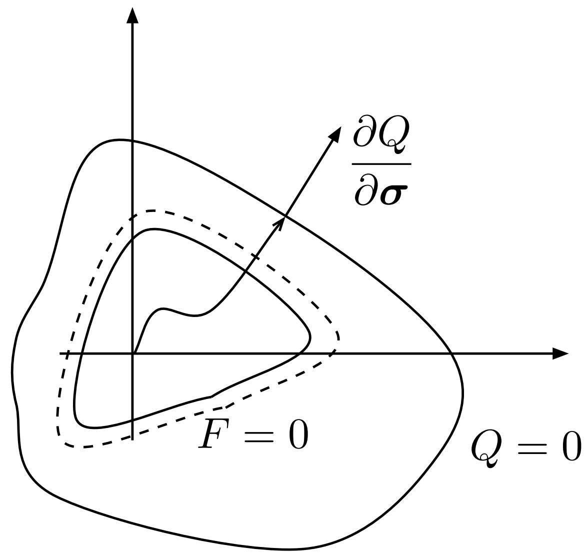

Figure 9: Yield function, hardening and the plastic potential in stress space

Number:	D3
Label:	D_YieldFunction
Symbol:	$F = F(\boldsymbol{\sigma}, \kappa)$
Type:	array1DT $\times \mathbb{R} \rightarrow \mathbb{R}$
Units:	–
Related Items:	D1, D4
Sources:	Perzyna (1966); Malvern (1969, pages 327–356); Mase (1970, pages 175–181); Zienkiewicz et al. (2005, pages 74–78)
Description:	The yield function defines a surface $F = 0$ in the six dimensional stress space, which can be visualized by looking at the sketch in Figure 9. Within this surface the material behaves as an elastic solid. Outside this surface the material is assumed to have yielded and thus must obey a different constitutive equation. When the material has yielded, which occurs when the stress path reaches the yield surface, as shown in Figure 9, the yield surface may change shape. This change in shape is caused by the strain hardening (or softening) of the material. The new yield surface is shown in Figure 9 as a dashed line. This behaviour is mathematically represented in the yield function by its dependence on the instantaneous values of the hardening parameter κ.
History:	Created – June 15, 2007

Number:	D4
Label:	D_HardeningParameter
Symbol:	$\kappa = \kappa(\boldsymbol{\epsilon}^{vp})$
Type:	array1DT $\rightarrow \mathbb{R}$
Units:	–
Related Items:	D2
Sources:	Malvern (1969, pages 363–373); Mase (1970, pages 182–183); Zienkiewicz et al. (2005, page 76)
Description:	The hardening parameter governs the size of the yield surface. It is a function of the accumulated viscoplastic strain tensor $\boldsymbol{\epsilon}^{vp}$. The accumulated viscoplastic strain is calculated from the rate of viscoplastic strain tensor $\dot{\boldsymbol{\epsilon}}^{vp}$, where the strain rate is defined as given in D2. The calculation of the accumulated viscoplastic strain from time 0 to time t is as follows: $\boldsymbol{\epsilon}^{vp} = \int_0^t \dot{\boldsymbol{\epsilon}}^{vp}(\tau) d\tau$.
History:	Created – Aug 16, 2007

Number:	D5
Label:	D_PlasticPotential
Symbol:	$Q = Q(\boldsymbol{\sigma})$
Type:	array1DT $\rightarrow \mathbb{R}$
Units:	–
Related Items:	D1
Sources:	Malvern (1969, pages 356–377); Mase (1970, page 182); Zienkiewicz et al. (2005, pages 74–78)
Description:	If a material has yielded the plastic potential function provides the direction of viscoplastic strain. As for the yield function, the potential function provides a surface ($Q = 0$) in stress space. The normal to this surface ($\frac{\partial Q}{\partial \boldsymbol{\sigma}}$), as shown in Figure 9, provides the direction of the viscoplastic strain increment. For many materials Q can be obtained from an isotropic expansion of the quasistatic yield surface. If this is the case, then the material is said to obey an associative flow rule.
History:	Created – June 15, 2007

Number:	D6
Label:	D_DescriptionOfMotion
Symbol:	$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$
Type:	vectorT $\times \mathbb{R} \rightarrow$ vectorT
Units:	Each component of the vectors $\mathbf{x}$ and $\mathbf{X}$ have the units of L, or in SI the units of meters (m).
Related Items:	–
Sources:	Long (1961, pages 48–51); Malvern (1969, pages 138–141)
Description:	The description of motion either focuses on a material particle or on a region in space. A detailed description is given below.
History:	Created – August 31, 2007

Detailed Description of Types of Descriptions of Motion

The descriptions of motion can be distinguished by considering the motion of an arbitrary material particle P corresponding to three different configurations over time, as shown in Figure 10 for a two-dimensional coordinate system. The motion of P can be described by a relation between the spatial position $\mathbf{x}$ and the initial coordinates $\mathbf{X}$ and time t; that is, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, with the independent variables being $\mathbf{X}$ and t. This equation expresses a material description of motion in a Lagrangian (L) formulation. In a Lagrangian analysis the initial configuration $\mathbf{X}$ provides a reference configuration to which all future variables are referred back to. Although the choice of a reference configuration is an arbitrary one, often the initial configuration of the body is selected for a Lagrangian analysis. If instead the reference configuration is continuously updated, then one has the updated Lagrangian (UL) formulation, in which $\mathbf{x} = \mathbf{x}(\mathbf{x}(\tau), t)$ with the independent variables being $\mathbf{x}(\tau)$ and t. In the UL approach all variables are expressed relative to the present configuration, at time τ, to find the state of the system in a future configuration at time $\tau + \Delta t$. For both the L and UL formulations one explicitly tracks the motion of the particles.

For the spatial, or the so-called Eulerian (E), formulation, time and the current location in space $\mathbf{x}$ are the independent variables. Since the focus is on a region in space, denoted by the control volume shown by the dashed line in Figure 10, particle P is not unique to the spatial point located at P'. Particle P, which is coincident with point P' at $t = \tau$, is one of many particles that pass through P'. It is for this reason that the kinematics of the spatial formulation are best expressed in terms of velocities and velocity gradients rather than displacements and displacement gradients.

3.3 Goal Statement

The family of material models has one common goal, as shown in the table below. As for the data definition tables, the goal is assigned a unique number and label and the table shows fields for the description of the goal, related items and history.

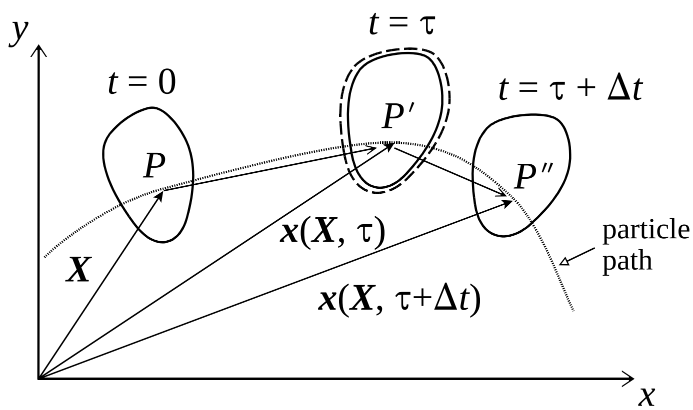

Figure 10: Different descriptions of motion

In this case the related item is the theoretical model. Changes in the theoretical model, potentially caused by changes in the assumptions it is based on, may cause the theoretical model to no longer satisfy the goal statement.

Number:	G1
Label:	G_StressDetermination
Description:	Given the initial stress and the deformation history of a material particle, determine the stress within the material particle.
Related Items:	T1
History:	Created – June 8, 2007

3.4 Assumptions

The assumptions documented in this section follow a template similar to that adopted for the data definitions in Section 3.2. As for the data definitions, the following fields are used: number, label, related items, description, source and history. The new fields introduced for the assumptions are as follows:

Equation: Some of the assumptions include an equation, when this makes the description more precise. For each equation the types of each of the terms is listed.

Rationale: This field justifies the appropriateness of the assumption within the context of the current family. If changes in the assumptions are made in the future, it will be because the rationale is inadequate in some sense.

Number:	A1
Label:	A_ContinuumHypothesis
Related Items:	–
Equation:	–
Description:	The underlying molecular structure of matter is not considered and gaps and empty spaces within a material particle are ignored. The material is assumed to be continuous.
Rationale:	The continuum hypothesis allows for the definition of stress and strain at a point. Although not strictly true, the notion that matter is continuous fits with what one usually sees at the macroscopic scale. Even at small fractions of the scales that engineering problems typically deal with there is generally an enormous number of molecules, which makes the averages of the physical properties stable. As the final, and most important justification, the theories built using the continuum hypothesis often provide quantitative predictions that agree closely with experimental data.
Source:	Long (1961, 33–34); Malvern (1969, pages 1–2)
History:	Created – Aug 23, 2007

Number:	A2
Label:	A_CauchyStress
Related Items:	D1
Equation:	–
Description:	The stress measure used in the description of the material behaviour is the Cauchy stress, which is the stress defined relative to the current deformed configuration at the spatial point $\mathbf{x}$. This choice is made rather than use a Piola-Kirchoff stress tensors, for which the stress is instead defined using a material point $\mathbf{X}$ in a reference state other than the current configuration. This reference state is usually the initial undeformed configuration, but the choice of reference is actually an arbitrary one, as long as the reference remains fixed.
Rationale:	The Cauchy stress is a suitable measure to use with the natural strain rate tensor that is being employed, since both are defined relative to the current deformed configuration. Moreover, the Cauchy stress tensor and the natural strain rate tensor, which is equivalent to the rate of deformation tensor, are conjugate variables in the energy sense (Malvern, 1969, page 232). Another argument in favour of the Cauchy stress is that it is by far the most commonly used stress measure in engineering applications.
Source:	Malvern (1969, pages 220–224)
History:	Created – Aug 23, 2007

Number:	A3
Label:	A_DeformationHistory
Related Items:	D2
Equation:	–
Description:	The deformation history of the material particle will be given as the history of the natural strain rate tensor. There are many other measures of deformation available, such as the Cauchy deformation tensor, Green deformation tensor, Eulerian strain tensor, Lagrangian strain tensor, but these measures will not be used within the current material model.
Rationale:	The natural strain rate tensor was selected as it is usually employed in the viscoplastic flow theory used here. The other measures of deformation would have to be manipulated to fit with the constitutive equation. In addition, the natural strain measure can lead to more rational results for large deformation than the results obtained using other strain measures because the natural strain more effectively removes the influence of rigid body rotation (Kato, 2006).
Source:	Malvern (1969, pages 150–170); Mase (1970, pages 77–109)
History:	Created – Aug 23, 2007

Number:	A4
Label:	A_NoDistribMoments
Related Items:	–
Equation:	–
Description:	There is an absence of distributed moments or couples stresses, which allows for a symmetric stress tensor.
Rationale:	This is a common assumption in continuum mechanics. The theory becomes much more complex without this assumption. The success of the assumption is born out by the success of theories built using it.
Source:	Mase (1970, pages 48–49)
History:	Created – Aug 17, 2007

Number:	A5
Label:	A_SmallDefGradients
Related Items:	–
Equation:	–
Description:	The material deformation gradient is assumed to be small, so that the strain rate tensor may be simplified.
Rationale:	Experience has shown that within a body, displacements tend to change gradually.
Source:	Mase (1970, pages 83)
History:	Created – Aug 23, 2007

Number:	A6
Label:	A_CartesianCoord
Related Items:	–
Equation:	–
Description:	A rectangular Cartesian coordinate system will be used to describe the stress, deformation and position of the material particle. That is, all vectors and tensors are represented using orthonormal unit base vectors. Other bases that could potentially have been included are the cylindrical and spherical coordinate systems.
Rationale:	The problems in continuum and computational mechanics can be solved using any coordinate system, as this is an arbitrary choice. As the rectangular Cartesian coordinate system is the one most frequently encountered in practice, this is the system adopted here.
Source:	Malvern (1969, pages 569–672) presents alternatives to the Cartesian coordinate system.
History:	Created – Sept 23, 2007

Number:	A7
Label:	A_Isotropic
Related Items:	–
Equation:	–
Description:	The material properties are independent of orientation; that is, the constitutive equation is unchanged in form if the reference axes are given any rigid rotation, reflection in a plane, or reflection in a point.
Rationale:	Many materials, such as metals, polymers and soils, have an internal structure that is symmetric about any plane. If this is not the case, such as for wood, then the constitutive equation is not isotropic.
Source:	Malvern (1969, pages 285)
History:	Created – Aug 23, 2007

Number:	A8
Label:	A_Isothermal
Related Items:	–
Equation:	–
Description:	The material properties are independent of temperature; that is, the constitutive equation does not account for changes in the temperature of the body.
Rationale:	In many engineering applications the temperature will change very little over time, or from point to point within the body; therefore, assuming that the body is isothermal is often a good approximation of reality.
Source:	–
History:	Created – Aug 23, 2007

Number:	A9
Label:	A_AdditivityPostulate
Related Items:	D2
Equation:	$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^{vp}$ with the following types and units $\dot{\boldsymbol{\epsilon}}$: array1DT (1/t) (1/s) $\dot{\boldsymbol{\epsilon}}^e$: array1DT (1/t) (1/s) $\dot{\boldsymbol{\epsilon}}^{vp}$: array1DT (1/t) (1/s)
Description:	The total strain rate ($\dot{\boldsymbol{\epsilon}}$) is assumed to decompose into elastic ($\dot{\boldsymbol{\epsilon}}^e$) and viscoplastic ($\dot{\boldsymbol{\epsilon}}^{vp}$) strain rates.
Rationale:	This is a standard assumption for elastoplastic and elastoviscoplastic materials. The appropriateness of this assumption is born out by the success of theories built upon it.
Source:	Malvern (1969, page 339); Mase (1970, page 181)
History:	Created – June 11, 2007

Number:	A10
Label:	A_ElasticConstit
Related Items:	D1, D2, A6, A8, A7
Equation:	$\dot{\boldsymbol{\sigma}} = \mathbf{D}\dot{\boldsymbol{\epsilon}}^e$ with $\mathbf{D} = \chi \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$ where $\chi = \frac{E}{(1+\nu)(1-2\nu)}$ The following types, and where appropriate units, are used: $\dot{\boldsymbol{\sigma}}$: array1DT (StressU/t) (Pa/s) $\mathbf{D}$: array2DT (StressU) (Pa) $\dot{\boldsymbol{\epsilon}}^e$: array1DT (1/t) (1/s) $E : \mathbb{R}^+$ (StressU) (Pa) ν : poissonT (dimensionless)
Description:	The rate of change of stress ($\dot{\boldsymbol{\sigma}}$) is assumed to linearly depend on the rate of natural elastic strain ($\dot{\boldsymbol{\epsilon}}^e$). The matrix **D** is known as the elastic constitutive matrix. The structure of this matrix is a consequence of the ordering of the stress and strain components and of the assumed isothermal conditions (A8), isotropic material (A7) and Cartesian. E is known as Young's Modulus and ν as Poisson's ratio.
Rationale:	Experimental evidence suggests that many materials behave as linear elastic materials when the strains are small. The structure of **D** comes from the fact that a linear elastic material is defined by a strain energy function and for this function to exist the relation between stress and strain should possess symmetry. The structure is further influenced, to the point where only two independent material properties are required, by the needs of an isotropic material. Details on the derivation of **D** can be found in Malvern (1969, pages 278–290).
Source:	Malvern (1969, page 281); Mase (1970, page 143); Zienkiewicz and Taylor (2005)
History:	Created – June 11, 2007

Number:	A11
Label:	A_PerzynaConstit
Related Items:	D2, D3, D5, A7, A8
Equation:	$\dot{\boldsymbol{\epsilon}}^{vp} = \gamma < \phi(F(\boldsymbol{\sigma}, \kappa) > \frac{\partial Q(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$ with $$< \phi(F) >= \begin{cases} \phi(F) & \text{if} \quad F > 0 \\ 0 & \text{if} \quad F \leq 0 \end{cases}$$ The following types and units are used: $\dot{\boldsymbol{\epsilon}}^{vp}$: array1DT (1/t) (1/s) $\gamma : \mathbb{R}^{+}$ $\phi : \mathbb{R} \rightarrow \mathbb{R}$ $F(\boldsymbol{\sigma}, \kappa)$: array1DT $\times \mathbb{R} \rightarrow \mathbb{R}$ $\frac{\partial Q(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$: array1DT $\boldsymbol{\sigma}$: array1DT (StressU) (Pa) $\kappa : \mathbb{R}$
Description:	The rate of viscoplastic strain ($\dot{\boldsymbol{\epsilon}}^{vp}$) is postulated to have the direction $\frac{\partial Q(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$ and the magnitude $\gamma < \phi(F(\boldsymbol{\sigma}, \kappa) >$ where γ is a fluidity parameter, F is the yield function, Q is the viscoplastic potential (also referred to as the dynamic loading surface) and $\phi(F)$ is some function of F.
Rationale:	This model of viscoplastic strain has been successfully applied to model many materials, including metals, plastics and geomaterials. This model also has the advantage that it includes other classes of constitutive behaviour, which include elastic, viscous, viscoelastic and plastic. Specific instances of the Perzyna constitutive equation are given in Section 6.
Source:	Perzyna (1966)
History:	Created – June 11, 2007

Number:	A12
Label:	A_DescriptionOfMotion
Related Items:	D6
Equation:	–
Description:	The description of motion is the UL formulation, but with Δt infinitesimal in size. The focus is on the strain rate $\dot{\boldsymbol{\epsilon}} = d\epsilon/dt$ rather than on the strain increments $d\boldsymbol{\epsilon}$.
Rationale:	The UL formulation is a natural choice for large deformation in solid mechanics because it is a material description of motion (as opposed to a spatial description), which means that attention is focused on the material particle, thus facilitating a natural and intuitive approach to writing the constitutive equation. The Lagrangian (L) approach is also a material description, but the book-keeping is much more difficult because quantities have to be defined with respect to a reference state, which could require complex large deformation tensors. Although the net result of a series of UL steps also allows for large deformation, for each individual step, the UL approach is simpler than the L description because only small deformations are considered in the UL.
History:	Created – Aug 30, 2007; Rationale added – Dec 18, 2007

3.5 Theoretical Model

The template for the table describing the theoretical model uses the fields of number, label, related items, description and history, as introduced in Section 3.2. In addition the table introduces the following fields:

Input: The input field consists of a list of the input variables and their types. Where appropriate, the units of the variables are listed as well.

Output: This field lists the output variable and its type. The units of the output variable is listed as well.

Derivation: The derivation explains how the theoretical model is derived from the assumptions on which it is based.

Number:	T1
Label:	T_ConstitEquation
Related Items:	A2, A3, A11, A9, A10, A12, V7
Input:	$\boldsymbol{\sigma}_0$: array1DT (StressU) (Pa) t_{begin} : $\mathbb{R}$ (t) (s) t_{end} : $\mathbb{R}$ (t) (s) $\dot{\boldsymbol{\epsilon}}(t) : \{t : \mathbb{R} \| t_{begin} \leq t \leq t_{end} : t\} \rightarrow$ array1DT (1/t) (1/s) mat_prop_val : string $\rightarrow \mathbb{R}$ $E : \mathbb{R}^+$ (StressU) (Pa) ν : poissonT (dimensionless)
Output:	$\boldsymbol{\sigma}(t) : \{t : \mathbb{R} \| t_{begin} \leq t \leq t_{end} : t\} \rightarrow$ array1DT such that $$\dot{\boldsymbol{\sigma}} = \mathbf{D}\left(\dot{\boldsymbol{\epsilon}} - \gamma < \phi(F(\boldsymbol{\sigma}, \kappa)) > \frac{\partial Q(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right) \text{ and } \boldsymbol{\sigma}(t_{begin}) = \boldsymbol{\sigma}_0$$ the components of $\boldsymbol{\sigma}$ have the units of StressU (Pa)
Derivation:	The governing differential equation is found by first solving for $\dot{\boldsymbol{\epsilon}}^e$ in A9 and then substituting the resulting expression into the elastic constitutive equation A10. The final form is found by substituting in the expression for $\dot{\boldsymbol{\epsilon}}^{vp}$ from A11.
Description:	The theoretical model is only completely defined once the associated variabilities (V7) that define the material have been set. Given the material properties, which always include E and ν, and the input parameters describing the deformation ($\dot{\boldsymbol{\epsilon}}(t)$) of the material particle over the relevant history from t_{begin} to t_{end} solve the governing differential equation for the stress history ($\boldsymbol{\sigma}(t)$) with the initial condition that the stress tensor $\boldsymbol{\sigma}(t_{begin}) = \boldsymbol{\sigma}_0$.
History:	Created – June 14, 2007

4 Variabilities

The tables used to present the variabilities borrow the following fields from the terminology definition template (Section 3.2): Number, Label, Related Items, Symbol, Type, Description and History. The variabilities also have an additional field for the binding time, where the binding time is the time in the software lifecycle when the variability is fixed. The binding time could be during specification of the requirements (specification time), or during building of the system (build time), or during execution of the system (run time). It is possible to have a mixture of binding times. For instance, a parameter of variation could have a binding time of "specification or build" to represent that the parameter could be set at specification time, or it could be postponed until the given family member is built. The presence of a DSL allows postponing the binding until build time. A DSL may in turn implicitly or explicitly specify that the binding will be postponed until run time.

Number:	V1
Label:	V_MatName
Related Items:	V2, V3, V4, V5, V6
Symbol:	*name*
Type:	string
Description:	The name of the material defined by the material related variabilities.
Binding Time:	Specification or Build
History:	Created – Sept 21, 2007

Number:	V2
Label:	V_YieldFunction
Related Items:	T1
Symbol:	$F = F(\boldsymbol{\sigma}, \kappa)$
Type:	array1DT $\times \mathbb{R} \to \mathbb{R}$
Description:	The yield function is one of the characteristics that distinguishes one member of the family of materials from another. The yield function may or may not include strain hardening (or softening) depending on whether the parameter κ is used or not. Many yield functions are written using the invariants of the stress tensor and of the deviatoric stress tensor. Also, many yield functions are convex surfaces in the stress space. The units of the yield function depend on the particular yield function. In many cases the units will be StressU (Pa) or StressU^2 (Pa^2). Example yield functions are given in Section 6.
Binding Time:	Specification or Build
History:	Created – June 12, 2007

Number:	V3
Label:	V_PlasticPotential
Related Items:	T1
Symbol:	$Q = Q(\boldsymbol{\sigma})$
Type:	array1DT $\rightarrow \mathbb{R}$
Description:	The plastic potential function is one of the characteristics that distinguished one member of the family of materials from another. In the case of associative materials, such as metals, the yield function and the plastic potential function will be the same. This is not the case for nonassociative flow materials, such as soils. The units of the plastic potential function depend on the particular function. In many cases the units will be StressU (Pa) or StressU^2 (Pa^2). Example plastic potential functions are given in Section 6.
Binding Time:	Specification or Build
History:	Created – Aug 24, 2007

Number:	V4
Label:	V_HardeningParameter
Related Items:	T1
Symbol:	$\kappa = \kappa(\boldsymbol{\epsilon}^{vp})$
Type:	array1DT $\rightarrow \mathbb{R}$
Description:	In the case of strain hardening (or softening) materials the yield function depends on the hardening parameter. In many cases the hardening parameter will be equal to the accumulated effective strain, which is the accumulation of the second invariant of the deviatoric strain tensor.
Binding Time:	Specification or Build
History:	Created – Aug 24, 2007

Number:	V5
Label:	V_Phi
Related Items:	T1
Symbol:	$\phi = \phi(F)$
Type:	$\mathbb{R} \rightarrow \mathbb{R}$
Description:	The function ϕ is used in the Perzyna constitutive equation. In many cases the function may be the identity function $\phi(F) = F$. Another common form of this function is the power-law form $\phi(F) = F^m$, where m is a material constant. Example ϕ functions can be found in Section 6.
Binding Time:	Specification or Build
History:	Created – Aug 24, 2007

Number:	V6
Label:	V_FluidityParameter
Related Items:	T1
Symbol:	γ
Type:	$\mathbb{R}^+$
Description:	A scalar constant that appears in the viscoplastic constitutive equation. For a viscous material the fluidity parameter may be related to the viscosity η as $\gamma = 1/(2\eta)$, where η will have units of StressU (Pa). The units for γ will need to be modified for other materials.
Binding Time:	Specification or Build
History:	Created – Aug 24, 2007

Number:	V7
Label:	V_MaterialProperties
Related Items:	V2, V3, V4, V5, V6
Symbol:	*mat_prop_names*
Type:	set of string
Description:	In addition to Young's modulus E and Poisson's ratio ν, each member of the family of materials will require additional material parameters. These parameters will be used to define the function F, Q and potentially κ. A constraint exists that the list of names provided by this variability has to match with the needs of the other variabilities.
Binding Time:	Specification or Build
History:	Created – Aug 24, 2007

Number:	V8
Label:	V_Description
Related Items:	V2, V3, V4, V5, V6
Symbol:	*descript*
Type:	string
Description:	A description of the material that is defined by the material related variabilities.
Binding Time:	Specification or Build
History:	Created – Sept 24, 2007

Number:	V9
Label:	V_StressState
Related Items:	D1
Symbol:	$\mathbf{s}_\sigma$
Type:	state_zeroT
Description:	This variability determines the state of stress for the material particle. If the ith entry in $\mathbf{s}_\sigma$ is *true*, then the ith entry in $\boldsymbol{\sigma}$ is known to always be zero for all uses of this particular family member. In general the stress state is 3D, with $\mathbf{s}_\sigma = \left\langle \textit{ false false false false false false } \right\rangle^T$. All of the entries in $\mathbf{s}_\sigma$ being *false* means that one does not know a priori if any of the values will be zero. Another example is the state of plane stress, where it is known in advance that $\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$. In this case $\mathbf{s}_\sigma = \left\langle \textit{ false false true false true true } \right\rangle^T$. A uniaxial state of stress in the x direction would require $\mathbf{s}_\sigma = \left\langle \textit{ false true true true true true } \right\rangle^T$. The entries in $\mathbf{s}_\sigma$ and $\mathbf{s}_\epsilon$ (Variability V10) need to be consistent between one another. A physical problem cannot require both the stress and the corresponding strain to be prescribed at the same time. The constraint may be expressed as follows: $$\forall i : \mathbb{N} \vert 1 \leq i \leq 6 : \neg(\mathbf{s}_\sigma[i] \wedge \mathbf{s}_\epsilon[i])$$
Binding Time:	Specification or Build
History:	Created – Sept 21, 2007

Number:	V10
Label:	V_StrainState
Related Items:	D2
Symbol:	$\mathbf{s}_\epsilon$
Type:	state_zeroT
Description:	This variability determines the state of strain for the material particle. If the ith entry in $\mathbf{s}_\epsilon$ is *true*, then the ith entry in $\boldsymbol{\epsilon}$ is known to always be zero for all uses of this particular family member. In general the strain state is 3D, with $\mathbf{s}_\epsilon = \left\langle \; \mathit{false} \quad \mathit{false} \quad \mathit{false} \quad \mathit{false} \quad \mathit{false} \quad \mathit{false} \; \right\rangle^T$. All of the entries in $\mathbf{s}_\epsilon$ being *false* means that one does not know a priori if any of the values will be zero. Another example is the state of plane strain, where it is known in advance that $\epsilon zz = \gamma_{yz} = \gamma_{xz} = 0$. In this case $\mathbf{s}_\epsilon = \left\langle \; \mathit{false} \quad \mathit{false} \quad \mathit{true} \quad \mathit{false} \quad \mathit{true} \quad \mathit{true} \; \right\rangle^T$. The entries in $\mathbf{s}_\sigma$ (Variability V9) and $\mathbf{s}_\epsilon$ need to be consistent between one another. A physical problem cannot require both the stress and the corresponding strain to be prescribed at the same time. The constraint may be expressed as follows: $$\forall i : \mathbb{N} \vert 1 \leq i \leq 6 : \neg(\mathbf{s}_\sigma[i] \wedge \mathbf{s}_\epsilon[i])$$
Binding Time:	Specification or Build
History:	Created – Sept 21, 2007

5 Dependence Graphs

Figure 11 shows the relationship between the common parts of the family of material models: the goal, the theoretical model, the data definitions and the assumptions. If an entry has a line between it and a higher entry, then this means that a change in the lower entry will likely require a change in the higher entry. This dependence graph summarizes the “Related Item” field in the tables given in the Commonalities Section of this report (Section 3). An example of a potential change would be to remove the assumption that the material is isothermal (A8). As the graph shows, removal of this assumption would mean changing the assumed form for the Perzyna constitutive equation (A11) and the elastic constitutive equation (A10). The change would involve making the material properties for the constitutive equation temperature dependent.

Figure 12 illustrates the dependence between the variabilities and the associated commonalities, and potentially between variabilities. As for the previous dependence graph (Figure 11), a line between a higher and a lower item shows that the higher item depends on the lower item. In this case if the lower item changes,

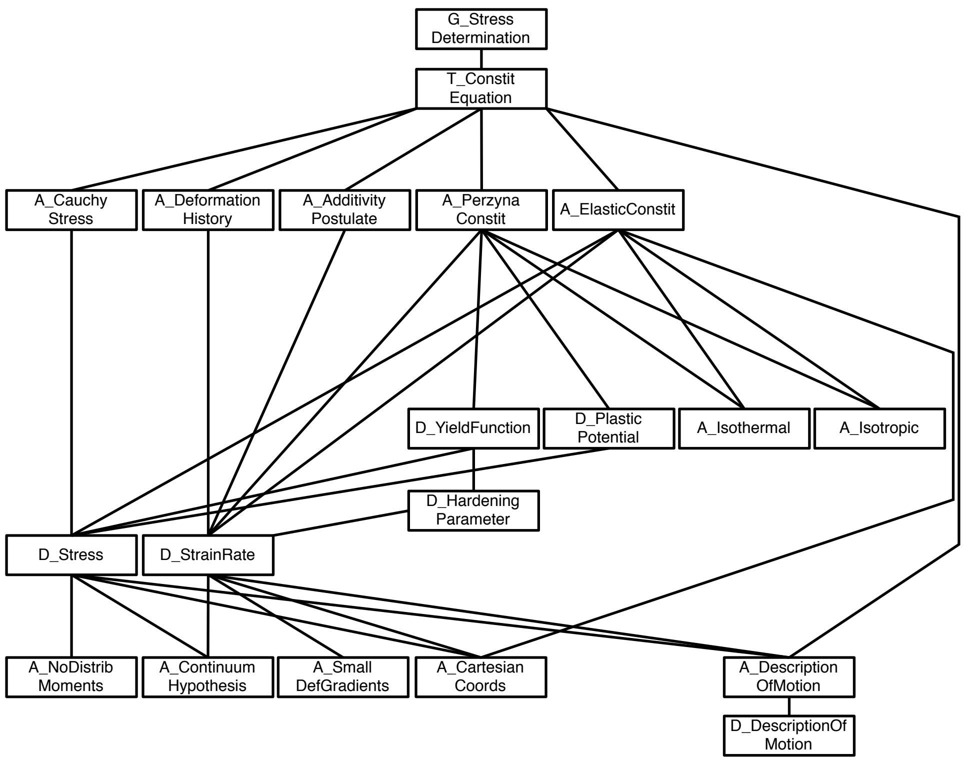

Figure 11: Dependence graph within the physical model (commonalities) of the family of material models

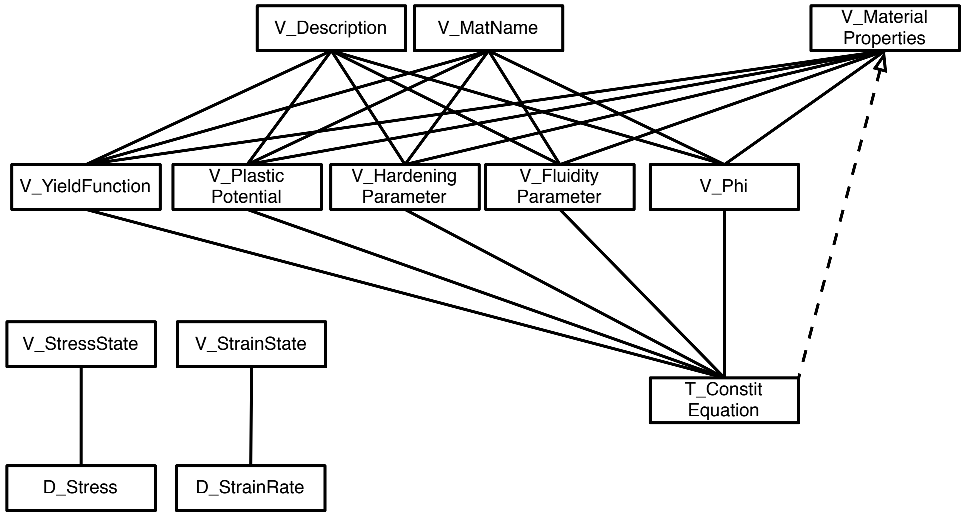

Figure 12: Dependence between the variabilities and the commonalities

then the associated variability could need to be modified. Several of the variabilities (V_Description, V_MatName and V_MaterialProperties) depend on other variabilities. This is to reflect the fact that the other variabilities need to be set before it makes sense to set the variabilities that describe the material and its material properties. Figure 12 shows a dashed arrow between T_ConstitEquation and V_MaterialProperties to indicate that expressing the theoretical model depends on the choice of material properties. This dependency occurs because the input to the theoretical model includes the values of the material properties, and the specific material properties are only known once the material model has been determined.

6 Sample Family Members

Several common members of the family of materials will be described below. In all cases it is assumed that a 3D state of stress and strain applies; that is,

$$\mathbf{s}_\sigma = \mathbf{s}_\epsilon = \left\langle \begin{array}{cccccc} \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} & \mathit{false} \end{array} \right\rangle^T \tag{28}$$

where $\mathbf{s}_\sigma$ and $\mathbf{s}_\epsilon$ are defined in variabilities V_StressState (V9) and V_StrainState (V10).

The equations that are used to define the sample family members use invariants of the stress and strain tensors and invariants of the deviatoric stress and strain tensors. Details on invariants and deviatoric tensors can be found in Malvern (1969) and Mase (1970). Only the information necessary to define the sample family members is summarized in this document.

The deviatoric tensors are found by subtracting the average of the normal components from each of the normal components. The deviatoric stress tensor $\mathbf{s}$ is defined as follows:

$$\mathbf{s} = \boldsymbol{\sigma} - \sigma_m \left\langle \begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right\rangle^T \text{ with } \sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{29}$$

where σ_m is 1/3 of the first invariant of the stress tensor ($\boldsymbol{\sigma}$). The term σ_m is known as the mean stress. The mean normal pressure p has the same magnitude as σ_m, but an opposite sign; that is, $\sigma_m = -p$.

The deviatoric strain tensor $\mathbf{e}$ is defined similarly to deviatoric stress tensor. The details are as follows:

$$\mathbf{e} = \boldsymbol{\epsilon} - \frac{\epsilon_v}{3} \left\langle \begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \end{array} \right\rangle^T \text{ with } \epsilon_v = (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \tag{30}$$

where ϵ_v is the first invariant of the strain tensor. This invariant is known as the volumetric strain.

An important invariant of the deviatoric stress tensor is its second invariant J_2, which is defined as follows:

$$J_2 = \frac{1}{2}\left(s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + 2(s_{xy}^2 + s_{yz}^2 + s_{xz}^2)\right) \tag{31}$$

A quantity related to this second invariant is the effective stress q (Malvern, 1969, page 364). The effective stress is defined so that for a uniaxial state of stress $q = |\sigma_{xx}|$, when the stress is along the x axis, with the corresponding change if the stress should be along another axis. The definition for q is as follows:

$$q = \sqrt{3J_2} \tag{32}$$

Similar to the definition of the second invariant of $\mathbf{s}$, the second invariant of the deviatoric strain tensor J_2^ϵ is defined as follows:

$$J_2^\epsilon = \frac{1}{2}\left(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)\right) \tag{33}$$

where γ is used for the shear terms instead of e to avoid confusion with textbook definitions where the e values associated with shear are half the value of the quantities used here. This difference is simply a consequence of the decision to use $\gamma = \epsilon/2$ for the shear terms (D2).

The corresponding strain measure to the effective stress is the effective strain (Malvern, 1969, page 364), which is defined as follows:

$$\epsilon_q = \sqrt{\frac{4}{3}J_2^\epsilon} \tag{34}$$

Each of the sample materials is summarized in a table. The rows of the tables correspond to the following variabilities:

V_YieldFunction (V2)	V_PlasticPotential (V3)	V_HardeningParameter (V4)
V_Phi (V5)	V_FluidityParameter (V6)	V_MaterialProperties (V7)
V_MatName (V1)	V_Description (V8)	

The variabilities for the state of stress and strain V_StressState (V9) and V_StrainState (V10) have been excluded because these variabilities have the same value for all of the examples, as shown at the beginning of this section.

The tables also include a row labelled "Source," which provides references that describe this particular family member. A row labelled "History" is provided, with the purpose of giving the history of the documentation of this particular family member within the CA document. Within these tables the notation $-$ is used to indicate that the value of the variability does not matter. That is, in these cases the variability can take on any syntactically correct value. This occurs when the variability is not actually used by the material model. For instance, the definition of the plastic potential Q does not matter for a purely elastic material, because it will never be necessary to determine the direction of the viscoplastic strain.

Number:	E1
Label:	E_Elastic
V1:	$name =$"Elastic"
V2:	$F = \infty$ (StressU) (Pa)
V3:	$Q = -$
V4:	$\kappa = -$
V5:	$\phi = -$
V6:	$\gamma = -$
V7:	$mat\_prop\_names = \emptyset$
V8:	$descript =$ "An elastic material is one that does not yield, so the yield function is set higher than any possible stress value. Since the material does not yield, there is not need to worry about the terms related to the viscoplastic strain rate. This material is linearly elastic because the stress depends linearly on the strain. In general, the linear elastic model is not valid for most materials above a strain of approximately 0.1 (or 10 %)."
Source:	Malvern (1969, pages 278–282); Mase (1970, pages 140–143); Beer and Johnston (1985)
History:	Created – Sept 21, 2007

Number:	E2
Label:	E_LinViscoElastic
V1:	*name* =“Linear Viscoelastic”
V2:	$F = q$ (StressU) (Pa)
V3:	$Q = q$ (StressU) (Pa)
V4:	$\kappa = -$
V5:	$\phi = F$ (StressU) (Pa)
V6:	$\gamma = 1/(2\eta)$ ($\mathrm{StressU}^{-1}\mathrm{t}^{-1}$) ($\mathrm{Pa}^{-1}\mathrm{s}^{-1}$)
V7:	$mat\_prop\_names = \{``\eta"\}$, where the type of η is $\mathbb{R}^+$ and the units are $\mathrm{StressU} \cdot \mathrm{t}$ ($\mathrm{Pa} \cdot \mathrm{s}$).
V8:	*descript* = “A linear viscoelastic material is linear in both the elastic contribution and the viscous contribution. This material is also known as a Maxwell fluid. This material will yield under any loading, since q will always be greater than zero. The material is analogous to a linear spring (elastic contribution) and a linear dashpot (viscous contribution) in series. For a spring and dashpot in series the total strain will be the sum of the strain in the spring and the strain in the dashpot, which corresponds to the Additivity Postulate (A9) assumed for the current family of material models. To clarify the behaviour of this material, one can consider the form of the constitutive equation for a uniaxial state of stress (V9). For a uniaxial state of stress the only nonzero stress for T1 is σ_{xx}, which is required to obey the following ordinary differential equation (ODE): $$\dot{\sigma}_{xx} = E(\dot{\epsilon}_{xx} - \frac{1}{2\eta}\sigma_{xx}) \text{ or } \lambda\dot{\sigma}_{xx} + \sigma_{xx} = 2\eta\dot{\epsilon}_{xx} \qquad (35)$$ where $\lambda = \frac{2\eta}{E}$ is known as the relaxation time. λ has type $\mathbb{R}^+$ and units of time t (seconds s in SI). For a constant rate of natural strain, the solution to this ODE is as follows: $$\sigma_{xx} = 2\eta\dot{\epsilon}_{xx}\left(1 - e^{-\frac{t}{\lambda}}\right) \qquad (36)$$ This equation shows that as λ gets smaller the material’s behaviour becomes more like that of a viscous fluid. Figure 5 shows what the stress strain curve looks like for the above closed-form solution for several different relaxation times.”
Source:	Malvern (1969, pages 313–317); Mase (1970, page 197); Joseph (1990)
History:	Created – Sept 21, 2007

Number:	E3
Label:	E_PowerLaw
V1:	*name* =“Power-law Viscoelastic”
V2:	$F = q$ (StressU) (Pa)
V3:	$Q = q$ (StressU) (Pa)
V4:	$\kappa = -$
V5:	$\phi = F^m$ (StressU^m) (Pa^m)
V6:	$\gamma = A$ (StressU^{-m}t^{-1}) (Pa^{-m}s^{-1})
V7:	*mat_prop_names* = {“m”, “A” }, where the types of the material properties are as follows: $$A : \mathbb{R}^+, m : \mathbb{R}^+ \quad (37)$$ There is likely an upper limit on the value for m, but at this time the value of this limit is unclear. With respect to units, m does not have units and A has units of (StressU^{-m}t^{-1}) (Pa^{-m}s^{-1}).
V8:	*descript* = “The power-law viscoelastic material is similar to the linear viscoelastic material mentioned previously, except that in this case the dashpot is nonlinear (for a dashpot oriented along the x axis $\sigma_{xx} = (\frac{1}{A}\dot{\epsilon}^v_{xx})^{\frac{1}{m}}$). When the nonlinear dashpot and linear spring are combined for a uniaxial state of stress (V9) the only nonzero stress for T1 is σ_{xx}, which will obey the following ordinary differential equation (ODE): $$\lambda\dot{\sigma}_{xx} + \sigma^m_{xx} = \frac{1}{A}\dot{\epsilon}_{xx} \quad (38)$$ where $\lambda = \frac{1}{AE}$. In the case that $m = 1$ and $A = 1/(2\eta)$, the linear viscoelastic material is recovered. When m is greater than 1 the material is known as shear thinning and when m is greater than 1 the material is known as shear thickening.”
Source:	Joseph (1990)
History:	Created – Sept 21, 2007

Number:	E4
Label:	E_StrainHardening
V1:	*name* =“Strain-Hardening Viscoelastic”
V2:	$F = q\kappa^{\frac{n-1}{m}}$ (StressU) (Pa)
V3:	$Q = q$ (StressU) (Pa)
V4:	$\kappa = \epsilon_q^{vp}$, which is the second invariant of the deviatoric viscoplastic strain tensor. (L/L) (m/m)
V5:	$\phi = F^{\frac{m}{n}}$ (StressU$^{\frac{m}{n}}$) (Pa$^{\frac{m}{n}}$)
V6:	$\gamma = nA^{\frac{1}{n}}$ (StressU^{-m}t^{-1}) (Pa^{-m}s^{-1})
V7:	*mat_prop_names* = {“A”, “m”, “n” }, where the type of the material properties are as follows: $$A : \mathbb{R}^+, m : \mathbb{R}^+, n : \mathbb{R}^+ \quad (39)$$ There is likely an upper limit on the values for m and n, but at this time the value of these limits is unclear. With respect to units, m and n do not have units and A has units of StressU^{-m}t^{-1} (Pa^{-m}s^{-1})
V8:	*descript* = “This constitutive equation combines the power-law viscoelastic material, described above, with a strain hardening (softening) material. A strain-hardening (softening) material is one where accumulated viscoplastic strain causes the material to be more difficult to deform (easier to deform). The strain hardening material will behave the same as the power-law viscoelastic material if $n = 1$ and it will behave like the linear viscoelastic material if $n = 1$, $m = 1$ and $A = \frac{1}{2\eta}$. In the case where $n < 1$ the material is strain hardening and when $n > 1$ the material is strain softening.”
Source:	Smith (2001)
History:	Created – Sept 21, 2007

Number:	E5
Label:	E_VonMises
V1:	*name* =“VonMises Viscoplastic”
V2:	$F = J_2 - k^2$ (StressU^2) (Pa^2)
V3:	$Q = J_2 - k^2$ (StressU^2) (Pa^2)
V4:	$\kappa = -$
V5:	$\phi = F$ (StressU^2) (Pa^2)
V6:	$\gamma = 1$ (StressU^{-2}t^{-1}) (Pa^{-2}s^{-1})
V7:	*mat_prop_names* = {“k” }, where the type of the material property k is as follows: $$k : \mathbb{R}^+ \tag{40}$$ The units of k are StressU (Pa).
V8:	*descript* = “The von Mises material yields when the distortion strain energy (related to J_2) reaches a maximum value. The value of k can be experimentally determined in two ways: 1. In pure shear experiment $k = \tau$, where τ is the observed yield shear stress. 2. In a uniaxial extension experiment $k = \frac{\sigma_y}{\sqrt{3}}$, where σ_y is the yield limit in pure tension. The von Mises material is an example of associative viscoplasticity, since $F = Q$.”
Source:	Malvern (1969, pages 337–338); Mase (1970, pages 178–179)
History:	Created – Sept 21, 2007

Number:	E6
Label:	E_DruckerPrager
V1:	$name$ =“Drucker-Prager Viscoplastic”
V2:	$F = \alpha\sigma_m + \sqrt{J_2} - k$ (StressU) (Pa)
V3:	$Q = \alpha\sigma_m + \sqrt{J_2} - k$ (StressU) (Pa)
V4:	$\kappa = -$
V5:	$\phi = F$ (StressU) (Pa)
V6:	$\gamma = 1$ (StressU^{-1}t^{-1}) (Pa^{-1}s^{-1})
V7:	mat_prop_names = {“α”, “k” }, where the type of the material properties are as follows: $$\alpha : \mathbb{R}, k : \mathbb{R} \quad (41)$$ Neither of the material properties has units.
V8:	$descript$ = “The Drucker-Prager material is an extension of the von Mises material. Unlike the von Mises material, the Drucker-Prager includes the first invariant of the stress tensor (σ_m). In some soil applications the yield function is rearranged so that the constants that are used are related to the cohesion and the friction angle.”
Source:	Pietruszczak (1996)
History:	Created – Sept 21, 2007

Number:	E7
Label:	E_MohrCoulomb
V1:	*name* =“Mohr Coulomb Viscoplastic”
V2:	$F = \frac{\sqrt{J_2}}{g(\theta)} - \eta\sigma_m - \mu$ (StressU) (Pa) where $g(\theta) = \frac{3-\sin\phi}{2\sqrt{3}\cos\theta - 2\sin\theta\sin\phi}, \eta = \frac{2\sqrt{3}\sin\phi}{3-\sin\phi}, \mu = \frac{2\sqrt{3}c\cos\phi}{3-\sin\phi}$
V3:	$Q = \frac{\sqrt{J_2}}{g(\theta)} - \eta^*\sigma_m - \mu^*$ (StressU) (Pa) where $g(\theta) = \frac{3-\sin\psi}{2\sqrt{3}\cos\theta - 2\sin\theta\sin\psi}, \eta = \frac{2\sqrt{3}\sin\psi}{3-\sin\psi}, \mu = \frac{2\sqrt{3}c\cos\psi}{3-\sin\psi}$
V4:	$\kappa = -$
V5:	$\phi = F$
V6:	$\gamma = 1$
V7:	*mat_prop_names* = { “c”, “ϕ”, “ψ” }, where the types of the material properties are as follows: $$c : \mathbb{R}, \phi : \text{angleT}, \psi : \text{angleT} \quad (42)$$ c is known as the cohesion and ϕ is the friction angle. The units for ϕ and ψ are radians.
V8:	*descript* = “The Mohr Coulomb criteria is most often used with soils or other frictional materials.”
Source:	Pietruszczak (1996)
History:	Created – Sept 21, 2007

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