

**SE 2AA4, CS 2ME3 (Introduction to Software  
Development)**

**Winter 2017**

# **08 Mathematics for MIS (H&S Ch. 3)**

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# Mathematics for MIS

- Administrative details
- Review of sets, relations and functions
- Review of logic
- Review of types, sets, sequence and tuples
- Multiple assignment statement
- Conditional rules
- Finite state machines

# Administrative Details

- Assignment 1
  - ▶ Files due by midnight January 28
  - ▶ E-mail partner files by January 28
  - ▶ Lab report due February 2
  - ▶ Using Python 2.7, doxygen, make, LaTeX, git
  - ▶ Make sure everything runs on mills
  - ▶ Use the folder structure and filenames given!

# Avenue Posts (or E-mail) Advice

- First sentence is the question
- Be as clear as possible on what you want
- Provide supporting information - OS, VM?, steps taken, error message
- Consider a screen shot of the error message
- Thank the people that helped you
- For E-mail add identifying information

# Mathematics for Module State Machines

- The material in this lecture should be review
- Shows the simple tools that can be used to build your MIS
- Shows the syntax we will use
- Follows
  - ▶ Hoffman and Strooper, Chapter 3
  - ▶ Gries and Schneider

# Sets, Relations and Functions

- A set is an unordered collection of elements
- A binary relation is a set of ordered pairs
- A function is a relation in which each element in the domain appears exactly once as the first component in the ordered pair

# Sets

- An element either belongs to a set or it does not
- $x \in S$  versus  $x \notin S$
- Defining a set
  - ▶ Enumerate  $\{x_1, x_2, x_3, \dots, x_n\}$
  - ▶ Logical condition (rule) from Gries and Schnieder notation  $\{x : t \mid R : E\}$ , where  $R$  is a predicate and  $E$  is an expression
  - ▶ An integer range  $[2..4] = \{2, 3, 4\}$ ,  $[7..4] = \{\}$
- Examples
  - ▶  $S = \{1, 7, 6\}$
  - ▶  $S = \{x : \mathbb{Z} \mid 1 \leq x \leq 4 : x\}$
  - ▶  $S = \{x : \mathbb{N} \mid 1 \leq 100 : x^2\}$
- Does  $\{1, 7, 6\} = \{7, 1, 6\}$ ?

# Sets

How would you write the set of points  $C$  inside a circle centered at the origin with a radius of  $r$ ?

- What is the type of each element in the set?
- What is the dummy variable?
- What is the range?



# Sets

How would you write the set of points  $C$  inside a circle centered at the origin with a radius of  $r$ ?

$$C(r) = \{x, y : \mathbb{R} | x^2 + y^2 \leq r^2 : \langle x, y \rangle\}$$

- What if the circle is centered at  $(x_c, y_c)$ ?

# Relations

- Let  $\langle x, y \rangle$  denote an ordered pair
  - ▶  $\text{dom}(R) = \{x | \langle x, y \rangle \in R\}$
  - ▶  $\text{ran}(R) = \{y | \langle x, y \rangle \in R\}$
- Defining a relation
  - ▶ Enumerate  $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 2, 3 \rangle\}$
  - ▶ Rule  $\{x, y : \mathbb{Z} | x < y : \langle x, y \rangle\}$

# Functions

- Let  $\langle x, y \rangle$  denote an ordered pair
- Each element of the domain is associated with a unique element of the range
- Defining a function
  - ▶ Enumerate  $\{\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle\}$
  - ▶ Rule  $\{x, y : \mathbb{Z} | y = x^2 : \langle x, y \rangle\}$
- Notation
  - ▶  $f(a) = b$  means  $\langle a, b \rangle \in f$
  - ▶  $f(x) = x^2$
  - ▶  $f : T_1 \rightarrow T_2$
  - ▶  $\{x_1, x_2, y : \mathbb{Z} | y = x_1 + x_2 : \langle \langle x_1, x_2 \rangle, y \rangle\}$
- Is  $\{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \langle 2, 3 \rangle\}$  a function?
- Is  $\{x, y : \mathbb{Z} | y^2 = x : \langle x, y \rangle\}$  a function?

# Logic

- A logical expression is a statement whose truth values can be determined ( $6 < 7?$ )
- Truth values are either *true* or *false*
- Complex expressions are formed from simpler ones using logical connectives ( $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\leftrightarrow$ )
- Truth tables
- Evaluation
  - ▶ Decreasing order of precedence:  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\leftrightarrow$
  - ▶ Evaluate from left to right
  - ▶ Use rules of boolean algebra

# Quantifiers

- Variables are often used inside logical expressions
- Variables have types
- A type is a set of values from which the variable can take its value
- Often quantify a logical expression over a given variable
  - ▶ Universal quantification
  - ▶ Existential quantification

# Quantifiers Continued

- Bound variables appear in the scope of the quantifier
- Free variables are not bound to any quantifier
- Free variables in an expression often mean that we cannot determine the truth value of the expression
- Gries and Schneider use the following notation  $(\forall x : t | R : P)$  where  $R$  is a predicate and  $P$  is a predicate
- $\forall$  is applied to the values  $P$  for all  $x$  in  $t$  for which range  $R$  is true

# Types, Sets, Sequence and Tuples

- A type is a set of values, so any precisely defined set is a type
- Primitive types are integer, boolean, character, string and real
- User-defined types
  - ▶ The set of values has to be given
  - ▶ Often use type constructors
- Useful type constructors
  - ▶ Set
  - ▶ Sequence
  - ▶ Tuple

# Types

- Specify the type of a variable
  - ▶  $x_1, x_2, \dots, x_n : T$
  - ▶  $x : \textit{integer}$  or  $x : \mathbb{Z}$
  - ▶  $a, b, c : \textit{string}$
- Type definition
  - ▶  $T = d$
  - ▶  $\textit{float} = \textit{real}$
  - ▶  $\textit{colour} = \{\textit{red}, \textit{white}, \textit{blue}\}$
  - ▶  $\textit{testtype} = \{\textit{uniaxial}, \textit{biaxial}, \textit{shear}\}$
  - ▶  $x : \textit{testtype}$
  - ▶  $\textit{motion}T = \{\textit{forward}, \textit{backward}, \textit{stop}\}$



# Primitive Types

- Integer

- ▶  $\{\dots - 2, -1, 0, 1, 2, \dots\}$
- ▶  $+, -, \times, /$
- ▶  $=, \neq$
- ▶  $<, \leq, \geq, >$

- Real

- ▶  $\{\textit{all real numbers}\}$
- ▶  $+, -, \times, /, \sin(), \cos(), \exp()$  etc.
- ▶  $=, \neq$
- ▶  $<, \leq, \geq, >$

# Primitive Types Continued

- Boolean type
  - ▶  $\{true, false\}$
  - ▶  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Char type
  - ▶ Set of ASCII characters
  - ▶ Character values appear in quotes *'a'*, *'b'*, *'c'*, etc.
  - ▶  $=, \neq$

# Primitive Types Continued

- String type

- ▶ All finite sequences of characters
- ▶ String constants are in double quotes *"abc"*
- ▶  $s[i..j]$  is the substring of  $s$  from position  $i$  to position  $j$
- ▶  $s_1||s_2$  concatenates strings  $s_1$  and  $s_2$
- ▶  $=, \neq$  for is equal and not equal
- ▶  $\in, \notin$  for is member and not a member
- ▶  $s[i]$  is the  $i$ th character of  $s$
- ▶  $|s|$  is the length of  $s$
- ▶ Positions in strings are zero relative

# Sets

- A set is an unordered collection of elements of the same type
- Declare a set of type  $T$  as *set of  $T$*
- Example
  - ▶  $T = \text{set of } \{red, green, blue\}$  defines type  $T$  as the power set of  $\{red, green, blue\}$
  - ▶  $x : \text{set of integer}$
- What are some possible values for  $x : \text{set of integer}$ ?

# Operations on Sets

- $\cup$  union
- $\cap$  intersection
- $-$  difference
- $\times$  Cartesian product
- $=, \neq$  equal, not equal
- $\in, \notin$  member, non-member
- $|s|$  size of set  $s$

# Sequences

- A sequence is an ordered collection of elements of the same type
  - ▶ Elements can occur more than once
  - ▶ Sometimes referred to as a list
  - ▶ Similar to an array
- Declare a sequence of type  $T$  by *sequence of  $T$*
- $\langle x_0, x_1, \dots, x_n \rangle$  for  $n \geq 0$  for a sequence with elements  $x_0, x_1, \dots, x_n$
- $\langle \rangle$  is the empty sequence
- Position in a sequence is zero relative

# Sequences Continued

- Examples
  - ▶  $T = \text{sequence of } \{\text{red}, \text{green}, \text{blue}\}$  defines the type  $T$  as the set of all sequences of elements from  $\{\text{red}, \text{green}, \text{blue}\}$
  - ▶  $x : \text{sequence of integer}$
- Fixed-length sequence of type  $T$  with length  $l$ 
  - ▶  $\text{sequence } [l] \text{ of } T$
  - ▶  $l$  is a positive integer
  - ▶  $\text{sequence } [l_1, l_2, \dots, l_n] \text{ of } T$  is a shorthand for  $\text{sequence } [l_1] \text{ of sequence } [l_2] \text{ of } \dots \text{sequence } [l_n] \text{ of } T$

# Operations on Sequences

- $s[i..j]$  is the subsequence of  $s$  from position  $i$  to position  $j$
- $s[i..j]$  is undefined if  $i \notin [0..|s| - 1] \vee j \notin [0..|s| - 1]$
- $s_1 || s_2$  concatenates sequences  $s_1$  and  $s_2$
- $=, \neq$  for is equal and not equal
- $\in, \notin$  for is member and not a member
- $s[i]$  is the  $i$ th element of  $s$
- $s[i]$  is undefined if  $i \notin [0..|s| - 1]$
- $|s|$  is the length of  $s$
- A string is a sequence of characters



# Tuples

- A tuple is a collection of elements of possibly different types
- Each tuple has one or more fields
- Each field has a unique identifier called the field name
- Similar to a record or a structure
- To declare a tuple use
  - ▶ *tuple of*  $(f_1 : T_1, f_2 : T_2, \dots, f_n : T_n)$  with  $n \geq 1$
  - ▶  $f_i$  is the name of the  $i$ th field
  - ▶  $T_i$  is the type of the  $i$ th field
  - ▶ *tuple of*  $(f_1, f_2, \dots, f_n : T)$  if all fields are of the same type

# Example Tuples

- Examples
  - ▶ `pair` = tuple of (`id`: integer, `val`: string)
  - ▶ `experimentT` = tuple of (`bcond`: `bcondT`, `control`: `controlT`)
  - ▶ `pointT` = tuple of (`x`: real, `y`: real)
- Define the value of a tuple by using an expression of the form
  - ▶  $\langle x_1, x_2, \dots, x_n \rangle$
  - ▶  $\langle 4, "cat" \rangle$  is a value of type `pair`

# Operations on Tuples

- $=, \neq$  equal, not equal
- $t.f$  is the value of field  $f$  of tuple  $t$

# Using Type Constructors

- $bcondT = \{uniaxial, biaxial, multiaxial, shear\}$
- $controlT = \{load\_controlled, displacement\_controlled\}$
- $experimentT = \text{tuple of } (b_{cond} : bcondT, control : controlT)$
- $experiment : experimentT$
- $directionT = \{clockwise, counterclockwise\}$
- $powerT = [MIN\_POWER \dots MAX\_POWER]$
- $motorT = \text{tuple of } (powerOn : Boolean, direction : directionT, powerLevel : powerT)$

# Multiple Assignment Statement

- $v_1, v_2, \dots, v_n := e_1, e_2, \dots, e_n$  with  $n \geq 1$
- The  $v_i$ s are distinct variables and each  $e_i$  is an expression of the same type as  $v_i$
- Compute the values of all the expression  $e_i$  and then assign these values simultaneously
- Example
  - ▶  $x, y := 0, 10$
  - ▶  $x, y := 10, x$
  - ▶  $x, y := y, x$
- Convenient for defining the meaning of pieces of code
- Use as a function on the state space of a program

# Uses of Conditional Rules

- To define the value of a function
- $\min(x, y) = (x \leq y \Rightarrow x \mid x > y \Rightarrow y)$
- To define the meaning of a program
  - ▶ If  $(x < y)$  then  $z := x$  else  $z := y$
  - ▶  $(x < y \Rightarrow z := x \mid x \geq y \Rightarrow z := y)$
  - ▶  $(x < y \Rightarrow x, y := x, y \mid x \geq y \Rightarrow x, y := y, x)$
- Conditional rules can be expressed in tables

# Finite State Machines

- A FSM is a tuple  $(S, s_0, I, O_E, O_O, T, E, C)$  where
- $S$  is a finite set of states
- $s_0$  is the initial state in  $S$  ( $s_0 \in S$ )
- $I$  is a finite set of inputs
- $T : S \times I \rightarrow S$  is the transition function
- $O_E$  is a finite set of event outputs
- $E : S \times I \rightarrow O_E$  is the event output
- $O_C$  is a finite set of condition outputs
- $C : S \rightarrow O_C$  is the condition output

# Homework Answer for `intersect(c)`

`out` : =?

- What is the type of `out`?
- How would you calculate the output (in words)?
- `intersect(c)` knows the centre and radius of the current object – how do we get the centre and radius of `c`?
- What is the type of the set of points inside a circle?



## Homework Answer for intersect(c)

out : =

$$\text{circlePts}(xc, yc, r) \cap \text{circlePts}(c.xc, c.yc, c.r) \neq \emptyset$$

$$\text{circlePts}(xc, yc, r): \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \text{set of } \mathbb{R}^2$$

$$\text{circlePts}(xc, yc, r)$$

$$\equiv \{x, y : \mathbb{R} \mid (x - xc)^2 + (y - yc)^2 \leq r^2 : \langle x, y \rangle\}$$

*(renamed getters to make syntax less awkward)*

## Homework Answer for intersect(c)

out :=

$$\begin{aligned} &(\exists \langle x, y \rangle : \mathbb{R}^2 \mid (x - xc)^2 + (y - yc)^2 \leq r^2 : \\ &(x - c.xc)^2 + (y - c.yc)^2 \leq (c.r)^2) \end{aligned}$$

*(renamed getters to make syntax less awkward)*

# References

- Hoffman and Strooper, *Software Design, Automated Testing and Maintenance*, International Thomson Computer Press, 1995.
- Gries and Schneider, *A Logical Approach to Discrete Math*, Springer, 1993.